Vzorkovník

$$\sum_{k=0}^{n} a^k = \frac{a^n - 1}{a - 1}$$

$$\sum_{k=0}^{n} \frac{1}{a^k} = \frac{a}{a-1} - \frac{1}{(a-1)a^n}$$

$$\sum_{k=0}^{n} k.a^k = \frac{na^{n+2} - (n+1)a^{n+1} + a}{(a-1)^2}$$

$$k.k! = (k+1)! - k!$$

$$\sum_{k=1}^{n} \frac{1}{k^m} = H_k^{(m)}$$

$$\sum_{k=1}^{n-1} H_k = \sum_{1 \leq j < k \leq n} \frac{1}{k-j} = nH_n - n$$

$$\sum_{k=0}^{n} k = \frac{n(n+1)}{2}$$

$$\sum_{k=0}^{n} k^2 = \square_n = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{k=0}^{n} k^3 = \binom{n+1}{2}^2$$

$$(n+1)H_{n+1} = nH_n + 1$$

$$F_1 + F_3 + \cdots + F_{2n+1} = F_{2n+2}$$

$$1 + F_2 + F_4 + \cdots + F_{2n} = F_{2n+1}$$

$$F_n^2 - F_{n+1}F_{n-1} = (-1)^{n+1}$$

$$\sum_{k=1}^{n} F_k^2 = F_n F_{n+1}$$

$$F_1F_2 + F_2F_3 + \cdots F_{2n-1}F_{2n} = F_{2n}^2$$

$$F_0 + F_1 + \cdots F_n = F_{n+2}$$

$$F_{n+m} = F_{m-1}F_n + F_mF_{n+1}$$

$$F_{n+1}F_{n+2} - F_nF_{n+3} = (-1)^n$$

$$F_{2n-1} = F_n^2 + F_{n-1}^2$$

$$F_{3n} = F_{n+1}^3 + F_n^3 - F_{n-1}^3$$

$$F_n^4 - F_{n-2}F_{n-1}F_{n+1}F_{n+2} = 1$$

$$\sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^k}{k}, \ (|x| < 1) = ln(1+x)$$

$$\lim_{n \to \infty} \left(a + \frac{1}{n}\right)^n = e$$

$$\sum_{k=0}^{\infty} F_k z^k = \frac{z}{1 - z - z^2}$$

$$\sum_{k=0}^{\infty} z^k = \frac{1}{1-z}$$

$$\sum_{k=0}^{\infty} \frac{1}{k!} z^k = e^z$$

$$\sum_{k=0}^{\infty} \binom{\alpha}{k} z^k = (1+z)^\alpha$$

$$\sum_{0 \leq k < m} \left\lfloor \frac{nk + x}{m} \right\rfloor = \sum_{0 \leq k < n} \left\lfloor \frac{mk + x}{n} \right\rfloor = a \left\lfloor \frac{x}{a} \right\rfloor + \frac{n-1}{2} n + \frac{a-m}{2}, \text{ kde } a = gcd(n, m)$$

Čebyševova sumačná nerovnosť: $\left(\sum_{k=1}^{n} a_k\right)\left(\sum_{k=1}^{n} b_k\right) \leq n \sum_{k=1}^{n} a_k b_k$, ak $\{a_i\}a\{b_i\}$ majú rovnaký charakter (nekles.-nerast.), inak $\geq$

Lagrangeova rovnosť: $\sum_{1 \leq j < k \leq n} (a_j b_k - a_k b_j)^2 = \left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} b_l^2\right) - \left(\sum_{k=1}^{n} a_k b_k\right)^2$

Perturbačná metóda: spravíme 2 spôsoby vyjadrenia – osamostatníme prvý, resp. posledný člen, uťapkáme a dáme vhodne do rovnosti

Integrovanie: prerobíme na integrál, ktorý vieme spočítať; musíme potom určiť chybu, o ktorú je ten integrál mimo, e.g. $\Box_n \to \int_0^n x^2 dx = \left[\frac{x^3}{3}\right]_0^n = \frac{n^3}{3}$, chyba je $E_n = \Box_n - \int_0^n x^2 dx = \Box_n - \frac{n^3}{3}$, dostaneme $E_n = E_{n-1} + n - \frac{1}{3} = \frac{n(n+1)}{2} - \frac{n}{3}$

Expand & contract: jednoduchú sumu rozložíme na zloženú, ktorá sa ale ľahšie poráta; e.g. $\Box_n = \sum_{0\le k\le n} k^2 = \sum_{0\le k\le n} \sum_{o<j\le k} k = \sum_{k=1}^n \sum_{j=1}^k k = \sum_{j=1}^n \sum_{k=j}^n k = \sum_{j=1}^n \left(\frac{n(n+1)}{2} - \frac{j(j+1)}{2}\right) = \frac{n^2(n+1)}{2} - \frac{1}{2}\sum_{1\le j\le n} j^2 + \frac{1}{2}\sum_{1\le j\le n} j = \dots$

Konečný kalkul: zopár definícií:

Def: *diferencia:* $\Delta f(x) = f(x+1) - f(x)$

Def: *klesajúca faktoriálna mocnina:* $x^{\underline{m}} = x(x-1)\cdots(x-m+1)$

Def: *rastúca faktoriálna mocnina:* $x^{\overline{m}} = x(x+1)\cdots(x+m-1)$

Def: *:* $x^{\underline{0}} = 1,\ x^{\underline{-1}} = \frac{1}{x+1},\ x^{\underline{-2}} = \frac{1}{(x+1)(x+2)}$

Potom platí:

$g(x) = \Delta f(x) \iff \sum g(x)\delta x = f(x) + c$, c môže byť správne naškálovaná funkcia

$$\sum_a^b g(x)\delta x = [f(x)]_a^b = f(b) - f(a),\ \Delta f(x) = g(x)$$

$$\sum_a^a g(x)\delta x = 0$$

$$\sum_a^{a+1} g(x)\delta x = f(a+1) - f(a) = \Delta f(a) = g(a)$$

$$\sum_a^{b+a} g(x)\delta x = g(b) + \sum_a^b g(x)\delta x$$

$$\sum_a^b g(x)\delta x = \sum_{k=a}^{b-1} g(k)$$

$$b < a,\ \sum_a^b g(x)\delta x = -\sum_b^a g(x)\delta x$$

$$\sum_a^b g(x)\delta x + \sum_b^c g(x)\delta x = \sum_a^c g(x)\delta x$$

$$\sum x^{\underline{m}}\delta x = \frac{x^{\underline{m+1}}}{m+1},\quad \Delta x^{\underline{m}} = \frac{x^{\underline{m+1}}}{m+1}$$

$$\sum v.\Delta u = u.v - \sum Eu.\Delta v,\ Ef(x) = f(x+1)$$

$$\Delta x^{\underline{m}} = mx^{\underline{m-1}}$$

$$\sum x^{\underline{m}}\delta x = \frac{x^{\underline{m+1}}}{m+1}$$

$$\Delta 2_x = 2^x = \sum 2^x \delta x$$

$$\Delta c^x = x^{x+1} - c^x = c^x(c-1) \to \sum_a^b c^x \delta x = \left[\frac{c^x}{c-1}\right]_a^b$$

$$\sum c^x \delta x = \frac{c^x}{c-1}$$

$$\Delta H_x = x^{\underline{-1}}$$

$$\Delta F_x = F_{x-1}$$

$$\sum F_x \delta x = F_{k+1}$$

$$\sum 2^{-x}\delta x = -2^{-k+1}$$

$(x+y)^{\underline{n}} = \sum_{k=0}^n \binom{n}{k} x^{\underline{k}} y^{\underline{n-k}}$, dôkaz sa robí indukciou, finta: $(x+y-m) = ((x-k) + (y-m+k))$

Príklad: $\Box_n = \sum_{k=0}^n k^2 = \left|k^{\underline{2}} = k(k-1) = k^2 - k\right| = \sum_0^{n+1} x^{\underline{2}} + x^{\underline{1}}\delta x = \left[\frac{x^{\underline{3}}}{3} + \frac{x^{\underline{2}}}{2}\right]_0^{n+1}$

20: Nech $y_n = x_1^n + x_2^n$, platí $p = x_1 + x_2$, $1 = x_1 x_2$. Potom všeobecný člen $y_n = p^n - \sum_{k=1}^{n/2}\binom{n}{k}p^{n-2k}$, e.g. $y_4 = p^4 - 4p^2 - 6$, $y_5 = p^5 - 5p^3 - 10p$, z čoho sa už ľahko uverí, že sú to všetko celé čísla, a teda aj y_{1995} a y_{1996}. Čo s deliteľnosťou, to netuším, ale z hentoho by už mohla pomaly aj vyplývať.

Binomické koeficienty a spriatelená chrobač

$$\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!} = \frac{n^{\underline{k}}}{k!}$$

$$\binom{r}{k} = \frac{r^{\underline{k}}}{k!} \text{ pre } k \geq 0, k \in Z; \qquad \binom{r}{k} = 0 \text{ pre } k < 0, k \in Z$$

$$\binom{n}{k} = \frac{n^{\underline{k}}}{k!} = \frac{n!}{k!(n-k)!}, \qquad n \geq k \geq 0$$

$$\binom{n}{k} = \binom{n}{n-k}, \qquad n \geq 0$$

$$\binom{r}{k} = \frac{r}{k}\binom{r-1}{k-1}, \qquad k \neq 0$$

$$k\binom{r}{k} = r\binom{r-1}{k-1}$$

$$(r-k)\binom{r}{k} = r\binom{r-1}{k}$$

$$\binom{r}{k} = \binom{r-1}{k} + \binom{r-1}{k-1}$$

$$\Delta\binom{x}{k} = \binom{x}{k-1}$$

$$\binom{r}{k} = (-1)^k\binom{k-1-r}{k}$$

$$\binom{r}{m}\binom{m}{k} = \binom{r}{k}\binom{r-k}{m-k}$$

$$\binom{2n}{n} = \binom{-1/2}{n}(-4)^n$$

$$\sum_{0\leq k\leq n}\binom{r+k}{k} = \binom{r+n+1}{n}$$

$$\sum_{0\leq k\leq n}\binom{k}{m} = \binom{n+1}{m+1}, \qquad m, n \geq 0$$

$$\sum\binom{x}{m}\delta x = \binom{x}{m+1} + c$$

$$\sum_{k\leq m}\binom{r}{k}(-1)^k = (-1)^m\binom{r-1}{m}$$

$$\sum_{0\leq k\leq n}\binom{r}{k}\binom{s}{n-k} = \binom{r+s}{n}$$

$$(x+y)^r = \sum_k \binom{r}{k} x^k y^{r-k}, \qquad r \geq 0 \text{ integer alebo } |x/y| < 1$$

$$(x+y+z)^m = \sum_{\substack{0\leq a,b,c\leq m\\ a+b+c=m}} \frac{(a+b+c)!}{a!\,b!\,c!} x^a y^b z^c = \sum \binom{a+b+c}{b+c}\binom{b+c}{c} x^a y^b z^c$$

Vandermondova konvolúcia

I.: $$\sum_k \binom{s}{m+k}\binom{r}{n-k} = \binom{r+s}{m+n}$$

II.: $$\sum_k \binom{l}{m+k}\binom{s}{n+k} = \binom{l+s}{l-m+n}, \qquad \text{int } l \geq 0$$

III.: $$\sum_k \binom{l}{m+k}\binom{s+k}{n}(-1)^k = (-1)^{l+m}\binom{s-m}{n-l}, \qquad \text{int } l \geq 0$$

IV.: $$\sum_{k\leq l} \binom{l-k}{m}\binom{s}{k-n}(-1)^k = (-1)^{l+m}\binom{s-m-1}{l-m-n}, \qquad \text{int } l, m, n \geq 0$$

V.: $$\sum_{0\leq k\leq l} \binom{l-k}{m}\binom{q+k}{n} = \binom{l+q+1}{m+n+1}, \qquad \text{int } l, m \geq 0, \ \text{int } n \geq q \geq 0$$