Combat 14.6.2002 - riešenia

1

$$\begin{aligned} S_{2m} &= \sum_{k=0}^{2m} (-1)^k k H_k = \sum_{k=1}^{m} [(2k)H_{2k} - (2k-1)H_{2k-1}] = \sum_{k=1}^{n} H_{2k+1} + m = \\ &= \frac{1}{2}\left[(2m+1)H_{2m} - \frac{1}{2}H_m - 2m\right] + m = \frac{1}{2}\left[(2m+1)H_{2m} - \frac{1}{2}H_m\right]. \end{aligned}$$

2

$$\begin{aligned} S_n &= \sum_k (-1)^k \binom{r-k}{n-k}\binom{r}{k} = \sum_k (-1)^{k+n-k}\binom{n-k-r+k-1}{n-k}\binom{r}{k} = \\ &= (-1)^n \sum_k \binom{n-r-1}{n-k}\binom{r}{k} = (-1)^n \binom{n-1}{n} = (n=0) \end{aligned}$$

3

$$\begin{aligned} S_n &= \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k(k+1)}\binom{n}{k} = \sum_{k=1}^{n}\left[\frac{(-1)^{k+1}}{k} - \frac{(-1)^{k+1}}{(k+1)}\right]\binom{n}{k} = \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k}\binom{n}{k} \\ &- \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k+1}\binom{n}{k} = H_n - \frac{n}{n+1} \end{aligned}$$

4

$$\sum_k \binom{l}{m+k}\binom{s}{n+k} = \sum_k \binom{l}{l-m-k}\binom{s}{n+k} = \binom{l+s}{l-m+n}$$

identita symetrie a Vandermondova konvolúcia, $m, n, k, l \in Z; l \geq 0$

5

$$\begin{aligned} S_n &= \sum_{k=1}^{n} \lfloor \log_2(2k+1) \rfloor \quad (\lfloor \log_2(2k+1) \rfloor = \lfloor \log_2(2k) \rfloor; \quad k > 0) \\ S_n &= \sum_{k=1}^{n} \lfloor \log_2(2k) \rfloor = (n+1)\cdot a - 2^{(a+1)} + 2 + n; \qquad a = \lfloor \log_2 n \rfloor \end{aligned}$$

6

$$\begin{aligned} S_n &= \sum_{k=1}^{n} \lfloor \log_2(k+1) \rfloor - \lceil \log_2 k \rceil = \lfloor \log_2(n+1) \rfloor + \sum_{k=1}^{n} \lfloor \log_2 k \rfloor - \lceil \log_2 k \rceil = \\ &= \lfloor \log_2(n+1) \rfloor + \sum_{k,j} ((k = 2^j) - 1)(1 \leq k \leq n) = \lfloor \log_2(n+1) \rfloor - n + \\ &+ \sum_{k,j} (k = 2^j)(1 \leq k \leq n) = \lfloor \log_2(n+1) \rfloor - n + \sum_j (0 \leq j \leq \lfloor \log_2 n \rfloor) = \\ &= \lfloor \log_2(n+1) \rfloor - n + \lfloor \log_2 n \rfloor + 1 \end{aligned}$$

7

$$
\begin{aligned}
S_{2m} &= \sum_{k=0}^{2m}(-1)^k k F_k = \sum_{k=1}^{m}(2kF_{2k}-(2k-1)F_{2k-1}) = \sum_{k=1}^{m}2k(F_{2k}-F_{2k-1}) + \sum_{k=0}^{m-1}F_{2k+1} = \\
&= 2\sum_{k=1}^{m}kF_{2k-2} + F_{2m} - 1 = F_{2m} - 1 + 2\sum_{k=0}^{m-1}(k+1)F_{2k} = F_{2m} - 1 + 2\sum_{k=0}^{m-1}kF_{2k} + \\
&+ 2\sum_{k=0}^{m-1}F_{2k} = (2m+2)F_{2m-1} - F_{2m} + 1
\end{aligned}
$$

8

$$
\begin{aligned}
S_n &= \sum_k \binom{n}{3k+1} = \frac{(1+1)^n + \omega_1^2(1+\omega_1)^n + \omega_2^2(1+\omega_2)^n}{3} = \\
&= \frac{1}{3}\left[2^n + 2(-1)^{n+1}\cdot\cos((2n-1)\pi/3)\right] \quad \omega_1 = e^{2\pi i/3}, \omega_2 = e^{4\pi i/3}
\end{aligned}
$$

9

$$a_n = 2a_{n-1} - a_{n-2} + (-1)^n; \quad n > 1; \quad a_0 = a_1 = 1$$

$$a_{n+2} - 2a_{n+1} + a_n = (-1)^n \qquad A(z) = \sum_k a_k z^k$$

$$\frac{A(z)-1-z}{z^2} - 2\frac{A(z)-1}{z} + A(z) = \frac{1}{1+z} \qquad |.z^2$$

$$A(z) = \frac{1}{(1-z)^2(1+z)} = \frac{1/2}{(1-z)^2} + \frac{1/4}{1-z} + \frac{1/4}{1+z}$$

$$a_n = \frac{n+1}{2} + \frac{1+(-1)^n}{4}$$

10

$$a_0 = 7, \quad 2a_n = na_{n-1} + 2n! \quad n > 0; \quad s_n = \frac{2^{n-1}}{n!}, \quad T_n = \frac{2^n}{n!}a_n$$

$$
\begin{aligned}
T_n &= T_{n-1} + 2^n, \quad n > 0, \quad T_0 = a_0 = 7 \\
T_n &= T_0 + \sum_{k=1}^{n} 2^k = 7 + 2^{n+1} - 2 = 2^{n+1} + 5 \\
a_n &= 2n! + \frac{5n!}{2^n}
\end{aligned}
$$