Exam answers

HERE ARE sketches of solutions to the midterm and final exam problems.

('M1' means midterm problem 1. Get it?)

M1 First we note that the recurrence has at most one solution: The addition formula defines $\left\lmoustache {n \atop k-1} \right\rmoustache$ for all n when $\left\lmoustache {n \atop k} \right\rmoustache$ is defined for all n, and it defines $\left\lmoustache {n \atop k} \right\rmoustache$ for all n when $\left\lmoustache {0 \atop k} \right\rmoustache$ is given and $\left\lmoustache {n \atop k-1} \right\rmoustache$ is defined for all n. Therefore if $f(n,k)$ is any function that satisfies the given recurrence equations, we must have $\left\lmoustache {n \atop k} \right\rmoustache = f(n,k)$ for all n and k.

Experimentation with small cases leads us to conjecture that

$$\left\lmoustache {n \atop k} \right\rmoustache \;=\; \binom{\lfloor n/2 \rfloor}{\lfloor k/2 \rfloor}[n \text{ odd or } k \text{ even}]\,.$$

Let $f(n,k)$ be the right-hand side of this equation. Setting $n=0$, we have

$$f(0,k) \;=\; \binom{\lfloor 0/2 \rfloor}{\lfloor k/2 \rfloor}[0 \text{ odd or } k \text{ even}] \;=\; \binom{0}{\lfloor k/2 \rfloor}[k \text{ even}] \;=\; [k=0]\,.$$

Setting $k=0$ clearly gives $f(n,0)=1$. So our proof will be complete if we can show that $f(n,k)$ satisfies the addition formula. If n is odd then

$$\begin{aligned}
&(-1)^k f(n-1,k) + f(n-1,k-1)\\
&\quad= (-1)^k\binom{\lfloor n/2 \rfloor}{\lfloor k/2 \rfloor}[k \text{ even}] + \binom{\lfloor n/2 \rfloor}{\lfloor (k-1)/2 \rfloor}[k \text{ odd}]\\
&\quad= \binom{\lfloor n/2 \rfloor}{\lfloor k/2 \rfloor} \;=\; f(n,k)\,.
\end{aligned}$$

And if n is even we have

$$\begin{aligned}
&(-1)^k f(n-1,k) + f(n-1,k-1)\\
&\quad= (-1)^k\binom{\lfloor n/2 \rfloor - 1}{\lfloor k/2 \rfloor} + \binom{\lfloor n/2 \rfloor - 1}{\lfloor (k-1)/2 \rfloor}\\
&\quad= \binom{\lfloor n/2 \rfloor}{\lfloor k/2 \rfloor}[k \text{ even}] \;=\; f(n,k)\,.
\end{aligned}$$

M2 (a) (Exercise 6.21 is similar.) Consider any sum $a/b + a'/b' = a''/b''$ where all three fractions are in lowest terms, and suppose that $2^k \backslash\backslash b$, $2^{k'} \backslash\backslash b'$,

$2^{k''}\backslash\backslash b''$. Then it's not difficult to prove that $k'' = \max(k, k')$ if $k \neq k'$, while $k'' < k$ if $k = k'$.

The answer is $\lfloor \lg n \rfloor$, since $1/2^{\lfloor \lg n \rfloor}$ is the unique fraction in the sum $\sum_{k=1}^{n} 1/k = H_n$ whose denominator is divisible by this many 2s.

(b) In binary notation we have $m = (\alpha 0 \beta)_2$ and $n = (\alpha 1 \gamma)_2$ for some bit-strings α, β, γ, where β and γ have the same length. Hence the highest power of 2 that occurs in the fractions $\frac{1}{m+1} + \cdots + \frac{1}{n}$ occurs uniquely in the term $\frac{1}{k}$ where $k = (\alpha 10 \ldots 0)_2$. This power is $2^{\lfloor \lg(m \oplus n) \rfloor}$.

(c) Clearly $m = n$ is a solution for all $n \geqslant 0$. Otherwise we may assume that $m < n$. Then $\frac{1}{m+1} + \cdots + \frac{1}{n} \equiv 0 \pmod 1$, so the largest power of 2 dividing the denominator of the sum $\frac{1}{m+1} + \cdots + \frac{1}{n}$ must be 2^0. Hence $\lfloor \lg(m \oplus n) \rfloor = 0$; hence $m \oplus n = 1$, and we must have $n = m + 1$, an odd number. But $\frac{1}{n} \equiv 0 \pmod 1$ iff $n = 1$. So the only solutions with $m \neq n$ are $\{m, n\} = \{0, 1\}$.

M3 We have $F_n = (\phi^n - \hat\phi^n)/\sqrt{5}$ and $\sum_{n \geqslant 1} z^n/n = -\ln(1-z)$; it follows that the stated sum is $1/\sqrt{5}$ times

$$\ln(1 - \hat\phi/2) - \ln(1 - \phi/2) \;=\; \ln \frac{2 - \hat\phi}{2 - \phi} \;=\; 4 \ln \phi\,,$$

since $2 - \hat\phi = 2 + \phi^{-1} = \phi^2$ and $2 - \phi = -\phi^{-2}$. Thus $C = 4/\sqrt{5}$.

M4 This problem takes us on a guided tour of the book; at each step there's only one "obvious" thing to do. First, if $p\backslash m$ the sum reduces to zero since $mj \bmod p = m^2 j \bmod p = 0$ for all j. Second, if $p\nmid m$ we can split the sum into two parts and then replace both $mj \bmod p$ and $m^2 j \bmod p$ by j, because these quantities run through the values $0 \leqslant j < p$ in some order. Third, we can evaluate the sum

$$\sum_{1 \leqslant k \leqslant j} \binom{k}{\lfloor \nu \rfloor} H_k \;=\; \binom{j+1}{\lceil \nu \rceil} \left(H_{j+1} - \frac{1}{\lceil \nu \rceil} \right), \qquad \nu = \ln(m+n)$$

using summation by parts and/or (6.70), since $H_0 = 0$ and ν is not an integer. Fourth, we can negate the upper index of $\binom{2n-x}{2n+1} = -\binom{x}{2n+1}$ and then use identity (3.4) to change ceiling to floor; the given sum has reduced to a telescoping series

$$\sum_{0 \leqslant j < p} \left(\left\lfloor \left(\frac{\binom{j+1}{\mu}\left(H_{j+1} - \frac{1}{\mu}\right)}{2n+1} \right) \right\rfloor - \left\lfloor \left(\frac{\binom{j}{\mu}\left(H_j - \frac{1}{\mu}\right)}{2n+1} \right) \right\rfloor \right)$$
$$= \left\lfloor \left(\frac{\binom{p}{\mu}\left(H_p - \frac{1}{\mu}\right)}{2n+1} \right) \right\rfloor, \qquad \mu = \lceil \ln(m+n) \rceil.$$

Multiply this last result by $[p\nmid m]$ to get the general answer.

M5 Assign new numbers as in 3.3. At time t, the second inspector is examining the building whose original number is the first element in the sequence

$$2t \bmod (2n+1)\,, \quad 2^2 t \bmod (2n+1)\,, \quad 2^3 t \bmod (2n+1)\,, \quad \ldots$$

that is $\leqslant n$. The other numbers in this sequence are the new numbers assigned to that building.

(a) When $n = 3m+1$ and $t = 2m+1$, this sequence begins with $4m+2$, then comes $(8m+4) \bmod (6m+3) = 2m+1$; the inspectors collide in building $2m+1$. (b) When $n = 7m+3$ and $t = 6m+3$, the sequence begins

$$\begin{aligned} 12m+6, \quad (24m+12) \bmod (14m+7) &= 10m+5, \\ (20m+10) \bmod (14m+7) &= 6m+3, \end{aligned}$$

hence they collide in building $6m+3$. (Notice that both cases (a) and (b) occur when $n = 10$.)

(c) In general suppose that t is the first element $\leqslant n$ in the stated sequence, and suppose t is the kth element. Then $k \geqslant 2$, and the sequence must have the form

$$2n+1-u, \quad 2n+1-2u, \quad 2n+1-4u, \quad \ldots, \quad 2n+1-2^{k-1}u$$

where $2t = 2n+1-u$ and $2n+1-2^{k-1}u = t$. These equations imply that $u = (2n+1)/(2^k-1)$ and $t = (2^{k-1}-1)u$. Hence there is at most one such sequence, and it exists iff $2n+1$ is divisible by 2^k-1. (This argument proves that the number of times the inspectors meet is exactly the number of divisors of $2n+1$ that have the form 2^k-1.)

Notice that if $2n+1$ is divisible by 2^k-1 and if p is a prime factor of k, then $2n+1$ is divisible by 2^p-1. So it suffices to test cases when k is prime. Finally, $2n+1$ is divisible by 2^p-1 iff $2n \equiv -1 \pmod{2^p-1}$ iff $n \equiv 2^{p-1}-1 \pmod{2^p-1}$.

F1 (a) By induction it's $1/x_1x_2\ldots x_n$, since this induction hypothesis tells us that the sub-sum when $\pi_n = k$ is

$$\frac{x_k}{x_1x_2\ldots x_n}\,\frac{1}{(x_1+\cdots+x_n)}\,.$$

(b) $\bigl(\sum_{k\in K} z_1^k/x_k\bigr)\ldots\bigl(\sum_{k\in K} z_n^k/x_k\bigr) = \sum_{k_1,\ldots,k_n\in K} z_1^{k_1}\ldots z_n^{k_n}/x_{k_1}\ldots x_{k_n}$, which we know from (a) is

$$\sum_{k_1,\ldots,k_n\in K}\;\sum_{\pi(n)} \frac{z_1^{k_1}\ldots z_n^{k_n}}{x_{k_{\pi_1}}(x_{k_{\pi_1}}+x_{k_{\pi_2}})\ldots(x_{k_{\pi_1}}+\cdots+x_{k_{\pi_n}})}$$

$$= \sum_{k_1,\ldots,k_n\in K}\sum_{\pi(n)} \frac{z_{\pi_1}^{k_{\pi_1}}\ldots z_{\pi_n}^{k_{\pi_n}}}{x_{k_1}(x_{k_1}+x_{k_2})\ldots(x_{k_1}+\cdots+x_{k_n})}\,.$$

Now set $z_1 = \cdots = z_n = z$ and divide by $n!$.

(c) Let $K = \{0, 1, 2, \ldots\}$ and $x_k = k!$; equate coefficients of z^m in the identity of part (b). (d) ${m \brace n}/m!$, by (7.49).

F2 (a) Alice has flipped exactly k heads with probability $\binom{m}{k}2^{-m}$, and Bill has flipped exactly k tails with probability $\binom{m}{m-k}2^{-m}$. So

Courtship rituals observed in this tribe of mathematicians were most peculiar.

$$P_m \;=\; 2^{-2m}\sum_{k=0}^{m}\binom{m}{k}\binom{m}{m-k} \;=\; 2^{-2m}\binom{2m}{m}\,,$$

which equals $(-1)^m\binom{-1/2}{m}$ by (5.37).

(b) According to (5.114) we have

$$\sum_{k<n+1}\binom{-1/2}{k}(-1)^k \;=\; (-1)^n\binom{-3/2}{n}\,,$$

which can also be written $\binom{n+1/2}{n}$. Subtract 1 because the term for $k = 0$ should not be included.

(c) One way is to write $\binom{n+1/2}{n} = (2n+1)\binom{2n}{n}/2^{2n}$ and apply Stirling's approximation to this formula. A slightly more difficult, but possibly more instructive, way is to proceed directly as follows: We have

$$\begin{aligned} \ln(n+\tfrac12)! \;&=\; (n+1)\ln(n+\tfrac12) - (n+\tfrac12) + \sigma + \tfrac{1}{12}n^{-1} + O(n^{-2})\,;\\ \ln n! \;&=\; (n+\tfrac12)\ln n - n + \sigma + \tfrac{1}{12}n^{-1} + O(n^{-2})\,;\\ \ln(n+\tfrac12) \;&=\; \ln n + \tfrac12 n^{-1} - \tfrac18 n^{-2} + O(n^{-3})\,. \end{aligned}$$

Hence $\ln(n+\frac12)! - \ln n! = \frac12\ln n + \frac38 n^{-1} + O(n^{-2})$. Taking the exponential of both sides yields

$$\frac{(n+\frac12)!}{n!} \;=\; n^{1/2} + \frac38 n^{-1/2} + O(n^{-3/2})\,,$$

a result that can also be obtained by using exercise 9.44 since $\left[{1/2 \atop -1/2}\right] = \binom{1/2}{2} = -\frac18$. Now divide by $\frac12!$ (see exercise 5.22), and get the answer:

$$2\sqrt{n/\pi} - 1 + 3/(4\sqrt{n\pi}) + O(n^{-3/2})\,.$$

(d) $P_{l,m} = P_l P_{m-l}$.

(e) Consider the random variable X_m = [they shake after mth flip]. Now $X = X_1 + \cdots + X_n$, and we can proceed as in the derivation of (8.24).

(f) We have

$$\begin{aligned}\sum_{1\leqslant l<m} P_{l,m} &= \sum_{1\leqslant l<m} (-1)^l\binom{-1/2}{l}(-1)^{m-l}\binom{-1/2}{m-l}\\ &= \sum_l (-1)^m\binom{-1/2}{l}\binom{-1/2}{m-l} - 2(-1)^m\binom{-1/2}{m}\\ &= 1-2P_m\end{aligned}$$

(see the derivation of (5.39)), hence $E(X^2)=\sum_{m=1}^{n}(2-3P_m)$.

(g) The variance is $E(X^2)-(EX)^2 = 2n-\binom{n+1/2}{n}^2-\binom{n+1/2}{n}+2$, so the standard deviation is $\sqrt{(2-4/\pi)n}+O(1)$.

(h) In this case $P_m=\binom{-1/2}{m}(-4pq)^m$, where $q=1-p$, so the average number of handshakes approaches $\sum_{m\geqslant1}P_m=(1-4pq)^{-1/2}-1$ (by the binomial theorem).

Let's resist the temptation to comment about disasters.

F3 (a) We can assume that $j=0$ and $k=n-1$, because the variables $x_j\ldots,x_k$ form a sequence of $k-j+1$ consecutive natural disasters. Then the numerator of $\Pr(x_0\geqslant x_{n-1}>x_1,\ldots,x_{n-2})$ is

$$\begin{aligned}&\int_0^1 dx_0\int_0^{x_0}dx_{n-1}\int_0^{x_{n-1}}dx_1\ldots\int_0^{x_{n-1}}dx_{n-2}\\ &\qquad = \int_0^1 dx_0\int_0^{x_0}dx_{n-1}\,x_{n-1}^{n-2} = \int_0^1 dx_0\,\frac{x_0^{n-1}}{n-1} = \frac{1}{n(n-1)}\,.\end{aligned}$$

(b) Let $\rho=1/(1+\epsilon)$. The probability P_k that x_k is a world record equals 1 minus the probability that it isn't a world record, namely

$$1-\sum_{j=0}^{k-1}\frac{\rho^{\binom{k-j+1}{2}}}{(k-j+1)(k-j)} = 1-\sum_{j=1}^{k}\frac{\rho^{\binom{j+1}{2}}}{j(j+1)}\,;$$

and $\rho^{\binom{j+1}{2}}=(1+\epsilon)^{-\binom{j+1}{2}}=1-\binom{j+1}{2}\epsilon+O(\epsilon^2)$. Thus, for fixed k,

$$P_k = 1-\sum_{j=1}^{k}\left(\frac{1}{j(j+1)}-\frac12\epsilon+O(\epsilon^2)\right) = \frac1k+\frac{k}{2}\epsilon+O(\epsilon^2)\,.$$

(c) And $M_n(\epsilon)=H_n+\frac14 n(n-1)\epsilon+O(\epsilon^2)$. [Note: This asymptotic value holds when ϵ is very very small, but it can be misleading when ϵ is larger because the error term is really $O(n^4\epsilon^2)$. Indeed, a closer analysis shows that when $(\ln n)^2/n^2\leqslant\epsilon\leqslant1$ we have $M_n(\epsilon)=\Theta(n\sqrt{\epsilon}\,)$. To prove this, we can break the sums into two ranges, using one estimate for $j\leqslant\sqrt{1/\epsilon}$ and another estimate for the larger values of j (when $\rho^{\binom{j+1}{2}}$ is getting small).]

F4 (a) Let α be such that $|S_{n-1} - \ln S_n| \leqslant \alpha$, and let β be any number greater than α. We will use the fact that $x \leqslant y - \beta$ implies $e^{x+\alpha} \leqslant e^y - \beta$ if y is sufficiently large, namely if $e^y \geqslant \beta/(1 - e^{\alpha-\beta})$. Take N large enough that $S_N \geqslant \ln\bigl(\beta/(1-e^{\alpha-\beta})\bigr)$ and define t by the relation $S_N = e\uparrow\uparrow(N+t)-\beta$. Then we can prove by induction that $S_n \leqslant e\uparrow\uparrow(n+t) - \beta$ for all $n \geqslant N$. Similarly, if we choose M and u so that $S_M = e\uparrow\uparrow(M+u) + \beta \geqslant \ln\bigl(\beta/(e^{\beta-\alpha} - 1)\bigr)$, then $S_n \geqslant e\uparrow\uparrow(n+u) + \beta$ for all $n \geqslant M$. Hence

$$e\uparrow\uparrow(n+u) \;\leqslant\; S_n \;\leqslant\; e\uparrow\uparrow(n+t)$$

for all large n; QED.

(b) Let $\ln A_n / \ln A_{n-1} = A_{n-2}(1 - \epsilon_n)$. The crude inequalities

$$\left(\frac{m-k}{k}\right)^k \;\leqslant\; \binom{m}{k} \;\leqslant\; m^k$$

tell us that $\epsilon_n \geqslant 0$ and that

$$\begin{aligned}\epsilon_n \;&\leqslant\; \frac{\ln A_{n-2}}{\ln A_{n-1}} - \frac{\ln(1 - A_{n-2}/A_{n-1})}{\ln A_{n-1}}\\ &=\; \frac{1}{A_{n-3}(1-\epsilon_{n-1})} + O\left(\frac{A_{n-2}}{A_{n-1}\ln A_{n-1}}\right).\end{aligned}$$

We can use bootstrapping to prove that ϵ_n is very small. First we prove inductively that $A_n \geqslant 2^n$ and that $A_n > 2A_{n-1}$ for all $n > 2$. Hence $\epsilon_n \leqslant 1/\bigl(A_{n-3}(1-\epsilon_{n-1})\bigr) + O(1/n)$, hence $\epsilon_n = O(1/n)$, hence $\epsilon_n = O(1/A_{n-3})$.

(The estimate $A_n \geqslant 2^n$ is "a bit conservative," but we need to start the proof somewhere. In fact, the sequence continues after A_1 with

$$\begin{aligned}A_2 &= 6\,,\\ A_3 &= 15\,,\\ A_4 &= 5005\,,\\ A_5 &= 2319752928920568707758603884212262733610400 0\,,\end{aligned}$$

and $A_6 \approx 8.2 \times 10^{200699}$ is too large to write down here.)

Now we have

$$\begin{aligned}\ln A_n \;&=\; (\ln A_1)\prod_{k=2}^{n}\frac{\ln A_k}{\ln A_{k-1}} \;=\; (\ln 4)\prod_{k=2}^{n} A_{k-2}(1-\epsilon_k)\\ &=\; \frac{C\prod_{k=0}^{n-2} A_k}{\prod_{k>n}(1-\epsilon_k)}\end{aligned}$$

where $C = (\ln 4)\prod_{k=2}^{\infty}(1-\epsilon_k)$. (This infinite product converges because we know that $\epsilon_k = O(2^{-k})$; in fact it converges very rapidly, and we have $C \approx 0.12824$.) Notice that $\ln\prod_{k>n}(1-\epsilon_k) = \sum_{k>n}\ln(1-\epsilon_k) = O\bigl(\sum_{k>n}\epsilon_k\bigr) = O(1/A_{n-2})$, hence

$$\begin{aligned}\ln A_n &= CA_0\ldots A_{n-2}\bigl(1+O(1/A_{n-2})\bigr)\,;\\ \ln\ln A_n &= \ln A_{n-2} + \ln\ln A_{n-1} + O(1/A_{n-2})\\ &= \ln A_{n-2} + O(n\log A_{n-3})\\ &= (\ln A_{n-2})\bigl(1+O(n/A_{n-4})\bigr)\,;\\ \ln\ln\ln A_n &= \ln\ln A_{n-2} + O(n/A_{n-4})\,.\end{aligned}$$

Merry Christmas and Happy New Year to all!

Now apply part (a) with $S_n = \ln\ln A_{\lfloor n/2\rfloor}$.