Final Exam

There are many students in this course, yet all exams need to be graded in 43 hours. PLEASE help by making your exam easy to grade. Think through your presentation before writing in the blue books. If you work out examples, just tabulate the results; don't give all the gory details. If you have significant intermediate results, label them for easy reference. Reserve your "stream of consciousness" abilities for another opportunity. However, don't be too sketchy; you need to convince the reader that you know what you are doing. Don't assume that the reader is clairvoyant; explain all steps. If you are proving something by contradiction, make your temporary assumptions stand out clearly.
—Friendly TAs

THIS IS A TAKE-HOME-and-open-everything-but-don't-get-help-from-other-people exam, due in room MJH 326 on Wednesday, December 14 before noon. There is no time limit. USE FIVE BLUE BOOKS, one for each of the five problems, SHOWING ALL YOUR WORK (so that partial credit can be given for incomplete answers). PLEASE SIGN YOUR NAME ON THE COVER OF EACH BLUE BOOK.

The problems have been designed so that you can almost always work each part independently, without having solved the previous parts. Therefore, don't give up on a problem just because you're stumped on part (a).

Problem 1: Broken records. (20 points)

Stanford's intramural committee is planning an n-day track meet, in which the same m events are to be run day after day. Assume that the n scores for each event will be distinct, and that each of the $n!$ permutations of the ranks of these scores will occur with equal probability, independent of the scores in all other events.

Find the mean and variance of the following three random variables:

a $A_{m,n}$ = total number of records broken during the entire meet. (On the first day, all m records are broken, by definition; on subsequent days, records are broken only in events where the score improved upon the best score of previous days. Hence about $\frac{1}{2}m$ records will be broken on the second day.)

b $B_{m,n}$ = total number of days on which records are broken in all m events.

c $C_{m,n}$ = total number of days on which at least one record is broken. In this case, express the mean and variance in terms of the function

$$S(m,n) \;=\; \sum_{k=1}^{n} \left(\frac{k-1}{k}\right)^{m} .$$

Problem 2: A problem suggested by Problem 1. (35 points)

Now we will find the asymptotic behavior of $S(cn,n)$ as $n \to \infty$, where S is the function defined in Problem 1 and c is a positive constant.

a Show that $\sum_{k=1}^{\lfloor n^{4/5} \rfloor} \left(\frac{k-1}{k}\right)^{cn} = O(n^{-2})$.

b Show that when $n^{4/5} \leqslant k \leqslant n$ we have

$$\left(\frac{k-1}{k}\right)^{cn} = \exp\left(-\frac{cn}{k} - \frac{cn}{2k^2} + O(n^{-7/5})\right).$$

c Prove that therefore

$$S(cn, n) = \sum_{k=\lfloor n^{4/5}\rfloor}^{n} \exp\left(-\frac{cn}{k} - \frac{cn}{2k^2}\right) + O(n^{-2/5}).$$

d Now use Euler's summation formula (9.67) with $m = 2$ to prove that

$$S(cn, n) = a(c)n + b(c) + O(n^{-1/5}),$$

where $a(c)$ and $b(c)$ can be expressed in terms of the function

$$E_1(c) = \int_c^\infty \frac{e^{-t}\,dt}{t}.$$

Problem 3: Another floored binomial sum. (30 points)

On the first problem of the midterm we derived an amazing identity involving the polynomials $t_n(r) = \binom{n+r}{\lfloor n/2\rfloor}$. Now let's prove that even more is true.

Please, don't remind me of the midterm.

a Find a closed form for the generating function

$$\sum_{n=0}^{\infty} t_n(r) z^n.$$

b Prove that the sum

$$\sum_{k=0}^{n} t_k(r) t_{n-k}(s-r)$$

does not depend on r.

c Find a closed form for the case $s = 0$, i.e., for the sum

$$\sum_{k=0}^{n} \binom{k+r}{\lfloor k/2\rfloor}\binom{n-k-r}{\lfloor (n-k)/2\rfloor}.$$

Problem 4: Social science. (30 points)

A set of students is called a *clique* if each student knows every other student in the set. A set of students is called a *claque* if none of them know each other.

According to this definition, a set is both a clique and a claque if it is empty or contains only one student.

a Suppose there are n students such that any pair know each other with probability p; these mutual-knowledge probabilities are independent of each other. (Thus, for example, the probability that Alice knows both Bill and Connie but that Bill doesn't know Connie is $p^2(1-p)$.) What is the expected number of cliques of size r? What is the expected number of claques of size s?

b Assume that the numbers n, r, and s have the property that every mutual-knowledge relation defined among n students necessarily contains either a clique of size r or a claque of size s. Prove that the sum of the two expected values in part (a) is $\geqslant 1$ for all p.

c Consider the set of all possible mutual-knowledge relations at universities that contain rF_{r-2} students, where F_k denotes a Fibonacci number. Show that, for all sufficiently large values of r, at least one of these possibilities contains no cliques of size r and no claques of size $2r$. *Hint:* Give a non-constructive proof, using the result of part (b).

Problem 5: A final recurrence. (35 points)

The numbers

n	1	2	3	4	5	6	7	8	9	10
a_n	1	2	7	32	178	1160	8653	72704	679798	7005632

Unfortunately, this sequence doesn't appear in Sloane's handbook [269].

are defined by the recurrence

$$a_1 = 1; \qquad a_{n+1} = (n+1)a_n + \sum_{k=1}^{n-1} a_k a_{n-k}, \qquad \text{for } n \geqslant 1.$$

The purpose of this problem is to determine the asymptotic behavior of a_n as $n \to \infty$.

a Show that

$$b_{n+1} \leqslant b_n + \frac{4b_n^2}{n(n+1)}, \qquad \text{if } b_n = a_n/n!.$$

Hint: Prove the inequality

$$\sum_{k=1}^{n-1} \binom{n}{k}^{-1} \leqslant \frac{4}{n}, \qquad \text{integer } n \geqslant 1.$$

b Prove that there exists a constant α such that

$$\lim_{n\to\infty} \frac{a_n}{n!} = \alpha.$$

Hint: Use the result of (a) to show that $b_n = O(\sqrt{n})$, then bootstrap and prove that $b_n = O(1)$.

c Prove that, in fact, $a_n = \alpha n! + O\big((n-1)!\big)$. *Hint:* THINK BIG, and make a conjecture about the approximate behavior of $a_n - \alpha n!$. Then apply the '$o(n) \Longrightarrow o(1)$' lemma in the class notes for November 30. You need not re-prove that lemma.

d (Bonus problem — work it only when you have finished the rest of the exam!) Show that the formal power series $\exp(a_1 z + \frac{1}{2}a_2 z^2 + \frac{1}{3}a_3 z^3 + \cdots)$ is hypergeometric, and use that fact to determine a closed form for the constant α. *Hint:* The numerical value of α is approximately 2.42819.

Oh well, this is better than another geography quiz.