In the previous lecture, we have seen how we can use complex analysis to find the exponential growth of the coefficients of a generating function. In this lecture, we will present some methods based on complex functions and their singularities, to derive more precise asymptotical estimations for the coefficients of generating functions. We state all theorems without proof. One who is interested in the proofs, can refer to [4].

Remark. The material of this lecture is taken from Flajolet & Sedgewick [3], and in particular Chapters 4 and 5 [4].

11.1 Expansions of rational functions

In this section, we use complex methods to find an exact asymptotic formula for the coefficients of a rational function. A rational function is a fraction of two polynomials. The following theorem yields an exact formula for the coefficients of a rational generating function.

Theorem 1 *(Expansion of rational functions) If $f(z)$ is a rational function that is analytic at zero and has poles at $\alpha_1, \alpha_2, \ldots, \alpha_m$, then there exists polynomials $P_j(x)$, $j = 1, 2, \ldots, m$, such that*

$$f_n = [z^n]f(z) = \sum_{j=1}^{m} P_j(n)\frac{1}{\alpha_j^n}.$$

Furthermore, the degree of $P_j(x)$ is equal to the order of the pole of f at α_j minus 1.

Example. Consider the following function:

$$f(z) = \frac{1}{(1-z^2)^3(1-z^3)^2(1-\frac{z^2}{2})}.$$

f has the following poles corresponding to each factor in its denominator:

$$\begin{array}{lcl} 1-z^2 & \rightarrow & 1, -1; \\ 1-z^3 & \rightarrow & 1, \omega, \omega^2, \quad \text{where } \omega = e^{2\pi i/3}; \\ 1-\frac{z^2}{2} & \rightarrow & \pm\sqrt{2}. \end{array}$$

Therefore, the set of poles of f is $\{1, -1, \omega, \omega^2, \sqrt{2}, -\sqrt{2}\}$.

The above theorem shows (and also, we observed in the previous lecture) that singularities that are closer to the origin are more important for the asymptotic behavior of f. For f, singularities at $1, -1, \omega, \omega^2$ all have modulus 1, and therefore are more important than $\pm\sqrt{2}$.

In addition, it can be proven that if we have multiple singularities with the same distance from the origin (i.e. with the same modulus), then the highest order singularity is the most important. The order of singularities of f are as follows:

$$1(5 \text{ times}), -1(3 \text{ times}), \omega(2 \text{ times}), \omega^2(2 \text{ times}), \sqrt{2}(1 \text{ time}), -\sqrt{2}(1 \text{ time}).$$

Therefore, the singularity at 1 is the dominant singularity of f. Note that:

If we are interested in the asymptotic behavior, it is sufficient to consider only the dominant singularity.

Thus, in order to find the asymptotic behavior of the coefficients of f, it is sufficient to find its behavior close to its pole at 1. So, we have to extract $(1-z)$ factors from the denominator of f and put $z=1$ in what remains:

$$f(z) = \frac{1}{[(1-z)(1+z)]^3[(1-z)(1+z+z^2)]^2(1-\frac{z^2}{2})} = \frac{1}{(1-z)^5} \times \frac{1}{(1+z)^3(1+z+z^2)^2(1-\frac{z^2}{2})}.$$

The value of $\frac{1}{(1+z)^3(1+z+z^2)^2(1-\frac{z^2}{2})}$ at $z=1$ is equal to $\frac{1}{2^3 3^2 \frac{1}{2}} = \frac{1}{36}$. Therefore, the behavior of f near 1 is similar to the behavior of $\frac{1}{36(1-z)^5}$ near 1. We can expand the latter function easily and compute its coefficients:

$$f_n = [z^n]f(z) \sim \frac{1}{36}[z^n]\frac{1}{(1-z)^5} = \frac{1}{36}\binom{n+1}{4} \sim \frac{n^4}{864}.$$

Thus, f_n is asymptotically equal to $\frac{n^4}{864}$.

Note that if there are two different singularities contributing with the asymptotic result, i.e. with the same modulus, and with the same order, then we must consider both of them in order to obtain the asymptotic behavior of f_n. In fact, we should calculate the asymptotic behavior for each of them, and add them up to find the result. (See [4] section 4.10, pp. 32–37.)

11.2 Expansion of meromorphic functions

In this section, we generalize Theorem 1 to a wider class of functions, namely meromorphic functions. Meromorphic functions were defined in the previous lecture. Equivalently, they can be defined as a quotient of two analytical functions. The following theorem yields an asymptotic estimate for the coefficients of meromorphic functions.

Theorem 2 *(Expansion of meromorphic functions) Let $f(z)$ be a function meromorphic for $|z| < R$ with poles $\alpha_1, \ldots, \alpha_m$, ($m$ is finite) and analytic for $|z| = R$ and at $z=0$. Then there exists polynomials $P_j(x)$, $j = 1, \ldots, m$, such that*

$$f_n = [z^n]f(z) = \sum_{j=1}^{m} P_j(n)\frac{1}{\alpha_j^n} + O\left(\frac{1}{R^n}\right).$$

Furthermore, the degree of $P_j(x)$ is equal to the order of α_j minus 1.

Example 1. Consider the generating function for the number of surjections:

$$S(z) = \frac{1}{2-e^z}.$$

It is clear that S is a meromorphic (but not rational) function. Singularities of S are at $\log 2 + 2\pi ik$ ($k = 0, \pm 1, \pm 2, \ldots$). The dominant singularity is at $z = \log 2$, and the next ones are at $\log 2 \pm 2\pi i$. Therefore, the dominant singularity of S has modulus $\log 2 = 0.69314\ldots$, and the next ones has modulus

$$\sqrt{(\log 2)^2 + (2\pi)^2} \cong \sqrt{39.95} \cong 6.32 > 6.$$

Now, we can use Theorem 2. For finding the behavior of S when $z \to \log 2$, we should extract the factor $(z - \log 2)$ from the denominator of S and calculate the limit of what remains, when $z \to \log 2$. After calculation, we obtain that:

$$S(z) \sim \frac{-1}{2(z - \log 2)}, \qquad \text{when } z \to \log 2.$$

Also,

$$[z^n]\frac{-1}{2(z - \log 2)} = [z^n]\frac{1}{2\log 2} \times \frac{1}{1 - \frac{z}{\log 2}} = \frac{1}{2\log 2} \times \frac{1}{(\log 2)^n} = \frac{1}{2(\log 2)^{n+1}}.$$

For $|z| < 6$, S is meromorphic and its only pole is at $\log 2$. Therefore, by Theorem 2,

$$S_n = \frac{1}{2(\log 2)^{n+1}} + O\left(\frac{1}{6^n}\right).$$

For instance, for $n = 10$, $\frac{1}{2(\log 2)^{n+1}} \cong 28.1766873361$, while $\frac{1}{6^n} \cong 1.65 \times 10^{-8}$. Therefore the above approximation is very accurate.

Example 2. (*The smallest cycle in a permutation*) Let us consider a random permutation of $\{1, 2, \ldots, n\}$, and decompose it into cycles. What is the expected value of the length of the smallest cycle?

To solve this problem, we can compute the probability that the length of the smallest cycle of a random permutation be greater than j, and take the summation of these probabilities over all values of $j \geq 0$. The value of this summation is equal to the expectation of the length of the smallest cycle. In other words, we use the fact that for every positive integer random variable X, $\text{Exp}(X) = \sum_{j\geq 0} j\Pr[X = j] = \sum_{j\geq 0}\Pr[X > j]$.

From previous lectures, we know that the probability that the minimum length of cycles in a random permutation of $\{1, \ldots, n\}$ is greater than k, is the coefficient of z^n in the following generating function

$$f(z) = \frac{e^{-z-\frac{z^2}{2}-\frac{z^3}{3}-\cdots-\frac{z^k}{k}}}{1 - z}.$$

We use Theorem 2 to find the asymptotic value of $[z^n]f(z)$. When $z \to 1$, and for a fixed k, we have:

$$f(z) \sim \frac{e^{-1-\frac{1}{2}-\frac{1}{3}-\cdots-\frac{1}{k}}}{1 - z} = \frac{e^{-H_k}}{1 - z}.$$

Thus, if n and k are large, we have the following approximation

$$f_n \sim e^{-H_k} \cong \frac{e^{-\gamma}}{k}.$$

Now, if we calculate the summation of this, over all k, we obtain the expectation. Hence, the expectation of the length of the smallest cycle is approximately $e^{-\gamma}\log n$. This result was first proven by Shepp and Lloyd [8]. A similar result for polynomials over finite fields was proven in [7].

Note that the above method is quite general and we can generalize the above result to obtain the expected value of the length of the second smallest cycle, the third smallest cycle, etc.

11.3 Singularity analysis

We have seen two theorems for extracting the asymptotic behavior of the coefficients of rational or meromorphic generating functions from their poles. But there are some generating functions which are not meromorphic. For example, the function

$$\frac{1}{(1-z)^\alpha}\left(\log\frac{1}{1-z}\right)^\beta$$

where α and β are arbitrary real or even complex numbers, is not a meromorphic function. The above function is called an *algebraic-logarithmic* function and occurs frequently in the generating functions of combinatorial objects.

There is a method due to Flajolet and Odlyzko [2] called *singularity analysis* which can be used to find asymptotics for many generating functions. There are also some other methods, such as Darboux method and Tauberian method which are well explained in a survey by Odlyzko in the Handbook of Combinatorics [6].

Singularity analysis is similar to the Darboux method [1]. The idea is to use the Cauchy coefficient theorem on some appropriate contour for integration. In singularity analysis method, Hankel contours are often used. Hankel contours, originated in analytic number theory, approach a singularity of the function in order to extract the behavior of its coefficients, from the behavior of the function near its singularity.

The singularity analysis method cannot be used in some cases. For example for functions which have an infinite number of singularities, or for fast growing functions, like $e^{\frac{z}{1-z}}$. For the latter case, we can apply the saddle point method (see Chapter 6 of [3] in [5]).

Now, for example, we mention the result obtained using the singularity method for the coefficients of $\frac{1}{(1-z)^\alpha}$ (α is an arbitrary number). For integer values of α, we know that:

$$\begin{aligned}[z^n]\frac{1}{(1-z)^\alpha} &= [z^n]\sum_{j=0}^{\infty}\binom{j+\alpha-1}{j}z^j = \frac{(n+1)(n+2)\cdots(n+\alpha-1)}{(\alpha-1)!} \\ &= \frac{n^{\alpha-1}}{(\alpha-1)!}\left(1+O\left(\frac{1}{n}\right)\right).\end{aligned}$$

For general α, there is a generalization of factorial, called *the Euler Gamma function*. The Gamma function is denoted by Γ and is defined as follows:

$$\Gamma(s) = \int_0^\infty e^{-t} t^{s-1} dt.$$

The Gamma function can be defined in the whole complex plane except for negative integers and zero. When α is a positive integer, we have $\Gamma(\alpha) = (\alpha - 1)!$. Therefore, the following theorem, is a natural generalization of the binomial coefficients formula for arbitrary α.

Theorem 3 *(Singularity analysis) Let α be a number not belonging to $\{0, -1, -2, \ldots\}$. The coefficient of z^n in $f(z) = \frac{1}{(1-z)^\alpha}$ admits for large n, a full asymptotic expansion in descending powers of n*

$$[z^n]f(z) \sim \frac{n^{\alpha-1}}{\Gamma(\alpha)} \left(1 + \sum_{k=1}^{\infty} \frac{C_k(\alpha)}{n^k}\right),$$

where C_k is a polynomial in α of degree $2k$ (for example $C_1 = \frac{\alpha(\alpha-1)}{2}$).

For example, for $\alpha = 1$, we obtain $[z^n]\frac{1}{1-z} \sim \frac{n^{1-1}}{\Gamma(1)} = 1$, which is accurate. For $\alpha = 2$, we obtain $[z^n]\frac{1}{(1-z)^2} \sim \frac{n^{2-1}}{\Gamma(2)} = n$, while the actual value of $[z^n]\frac{1}{(1-z)^2}$ is $n+1$ (of course, we are taking $n \to \infty$, so the result is accurate). For $\alpha = \frac{1}{2}$, we know that $[z^n]\frac{1}{\sqrt{1-z}}$ is equal to $\frac{1}{4^n}\binom{2n}{n}$. We have shown that the Catalan number $\frac{1}{n+1}\binom{2n}{n} \sim \frac{4^n}{n\sqrt{n\pi}}$. Theorem 3 gives the asymptotic estimation $\frac{n^{-1/2}}{\Gamma(1/2)}$, which is equal to $\frac{1}{\sqrt{n\pi}}$, since it can be proved that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

As another example, it can be proved using singularity method that

$$[z^n]\frac{1}{(1-z)^\alpha}\left(\log\frac{1}{1-z}\right)^\beta \sim \frac{n^{\alpha-1}}{\Gamma(\alpha)}(\log n)^\beta.$$

For instance, when $\alpha = \beta = 1$, we have $[z^n]\frac{1}{1-z}\log\frac{1}{1-z} \sim \log n$. We have known that this coefficient is equal to the harmonic number H_n.

In general, when we find an asymptotic for a generating function f, we encounter an equation like the following

$$f(z) = g(z) + O(h(z)),$$

but we are interested in the coefficients of $f(z)$. Can we deduce from the above equation that $[z^n]f(z) = [z^n]g(z) + O([z^n]h(z))$? In other words, can we say that $[z^n]O(h(z)) = O([z^n]h(z))$? This is not a trivial question at all. But there are some theorems called *transfer lemmas*, which do this for us. The proofs are based on estimates of Cauchy's coefficient formula using Hankel contours.

Here we summarize the process of finding an asymptotic estimation for the coefficients of a generating function: First, we find an asymptotic for our generating function in terms of a standard function. Then, we find the coefficients of the standard function. Finally, we estimate the error term using transfer lemmas. Therefore, we need two things:

1. A catalogue of exact asymptotic forms of coefficients of standard functions. Actually, it is sufficient to have a catalogue for functions with singularity at $z = 1$, because if f is singular at $z = \rho$, then $g(z) = f(z/\rho)$ has a singularity at $z = 1$ and $[z^n]f(z) = \rho^n [z^n]g(z)$.

2. For error term, we need transfer lemmas. For example, there is a lemma saying that for every complex number α not belonging to $\{0, -1, -2, \ldots\}$, we have:

$$\begin{array}{lcl} f(z) = O\left(\frac{1}{(1-z)^\alpha}\right) & \Longrightarrow & f_n = O(n^{\alpha-1}), \\ f(z) = o\left(\frac{1}{(1-z)^\alpha}\right) & \Longrightarrow & f_n = o(n^{\alpha-1}), \\ f(z) \sim \frac{1}{(1-z)^\alpha} & \Longrightarrow & f_n \sim \frac{n^{\alpha-1}}{\Gamma(\alpha)}. \end{array}$$

   This lemma is sufficient to prove many asymptotic results. Note that when $\alpha \in \{0, -1, -2, \ldots\}$, $[z^n]\frac{1}{(1-z)^\alpha}$ vanishes for large n.

References

[1] G. Darboux. Mémoires sur l'approximation des fonctions de trés-grands nombres, et sur une classe étendue de développements en série. *J. Math. Pures Appl.*, 4:5–56, 377–416, 1878.

[2] P. Flajolet and A. Odlyzko. Singularity analysis of generating functions. *SIAM Journal on Discrete Mathematics 3*, 2:216–240, 1990.

[3] P. Flajolet and R. Sedgewick. Analytic combinatorics. In preparation.

[4] P. Flajolet and R. Sedgewick. The average case analysis of algorithms: complex asymptotics and generating functions. Technical Report 2026, INRIA, Rocquencourt, Le Chesnay, France, September 1993.

[5] P. Flajolet and R. Sedgewick. The average case analysis of algorithms: saddle point asymptotics. Technical Report 2376, INRIA, Rocquencourt, Le Chesnay, France, 1994.

[6] A. Odlyzko. Asymptotic enumeration methods. In R. Graham, M. Grötschel, and L. Lovász, editors, *Handbook of Combinatorics*, volume 2, pages 1063–1229. Elsevier, 1995.

[7] D. Panario and B. Richmond. Analysis of Ben-Or's polynomial irreducibility test. *Random Struct. Alg.*, 13:439–456, 1998.

[8] L.A. Shepp and S.P. Lloyd. Ordered cycle lengths in a random permutation. *Trans. Amer. Math. Soc.*, 121:340–357, 1966.