3.1 Exponential Generating Functions

In the previous lecture we presented the Ordinary Generating Function (OGF). In this lecture we first introduce the *Exponential Generating Functions* (EGF). Similar results and methodology to the ones for OGF exist for this type of generating functions. Then, we start with our next topic: real asymptotics.

Remark. The material of this section is taken from Sedgewick and Flajolet [7], Chapter 3. Other sources and material for further study are [2, 3, 5, 6].

Definition 1 *For a sequence $a_0, a_1, \ldots, a_k, \ldots$, the function*

$$A(z) = \sum_{k\geq 0} a_k \frac{z^k}{k!}$$

is the Exponential Generating Function*(EGF) for the sequence. We have the following notation: $n![z^n]A(z) = a_n$, for the elements of the sequence.*

We first note that the EGF of a sequence $\{a_n\}$ is the OGF of the sequence $\{a_n/n!\}$. However, in some particular occasions it is better to deal with EGF directly. Suppose that a_n represents a count associated with a structure of n items, and assume that the items are *labelled* so that each one has a distinct identity. The factor $n!$ accounts for all the arrangements of the labelled items. These arrangements become indistinguishable if the items are *unlabelled.* When the labeling is relevant we use EGF, while when the labeling is not relevant we use OGF. We will come back to this several times in this course.

Examples of Exponential Generating Functions

(a) $1, 1, 1, \ldots$ $\qquad \sum_{k\geq 0} 1 \cdot \frac{z^k}{k!} = e^z.$

(b) $1, 2, 3, 4, \ldots, k \ldots$ $\qquad \sum_{k\geq 0} \frac{kz^k}{k!} = z\sum_{k\geq 1} \frac{z^{k-1}}{(k-1)!} = ze^z.$

(c) $0, 0, 1, 3, 6, \ldots, \binom{k}{2}, \ldots$ $\qquad \sum_{k\geq 2} \binom{k}{2}\frac{z^k}{k!} = \frac{1}{2}z^2\sum_{k\geq 2}\frac{z^{k-2}}{(k-2)!} = \frac{1}{2}z^2e^z.$

(d) $0, \ldots, 0, 1, m+1, \ldots, \binom{k}{m}, \ldots$ $\qquad \sum_{k\geq m} \binom{k}{m}\frac{z^k}{k!} = \frac{1}{m!}z^m\sum_{k\geq m}\frac{z^{k-m}}{(k-m)!} = \frac{1}{m!}z^me^z.$

(e) $1, c, c^2, c^3, \ldots, c^k, \ldots$ $\qquad \sum_{k\geq 0} c^k\frac{z^k}{k!} = \sum_{k\geq 0}\frac{(cz)^k}{k!} = e^{cz}.$

(f) $1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{k+1}, \ldots$ $\qquad \sum_{k\geq 0}\frac{1}{k+1}\frac{z^k}{k!} = \frac{1}{z}\sum_{k\geq 0}\frac{z^{k+1}}{(k+1)!} = \frac{e^z-1}{z}.$

(g) $1, 1, \frac{1}{2!}, \frac{1}{3!}, \ldots, \frac{1}{k!}, \ldots$ $\qquad \sum_{k\geq 0} k!\frac{z^k}{k!} = \frac{1}{1-z}.$

(h) $1, 0, 1, 0, \ldots$ $\qquad \sum_{k\geq 0}\frac{z^{2k}}{(2k)!} = \sum_{k\geq 0}\frac{1+(-1)^k}{2}\frac{z^k}{k!} =$
$\sum_{k\geq 0}\frac{z^k}{2k!} + \sum_{k\geq 0}\frac{(-z)^k}{2k!} = \frac{e^z+e^{-z}}{2}.$

Given two generating functions we can operate with them obtaining generating functions for other sequences. Some of the possible transformations to bear in mind are given by the following theorem.

Theorem 1 *If* $A(z) = \sum_{k\geq 0} a_k \frac{z^k}{k!}$ *and* $B(z) = \sum_{k\geq 0} b_k \frac{z^k}{k!}$ *are two exponential generating functions, then the below defined operations produce the following sequences:*

1. **Right-shift (integration):** $0, a_0, a_1, \ldots, a_{k-1}, \ldots$
   $\sum_{k\geq 1} a_{k-1} \frac{z^k}{k!} = \int_0^z A(u)du.$
2. **Left-shift (differentiation):** $a_1, a_2, \ldots, a_{k+1}, \ldots$
   $\sum_{k\geq 0} a_{k+1} \frac{z^k}{k!} = A'(z).$
3. **Index multiply:** $0, a_0, 2a_1, \ldots, ka_{k-1}, \ldots$
   $\sum_{k\geq 1} ka_{k-1} \frac{z^k}{k!} = zA(z)$
4. **Index divide:** $a_1, \frac{a_2}{2}, \ldots, \frac{a_{k+1}}{k+1}, \ldots$
   $\sum_{k\geq 0} \frac{a_{k+1}}{k+1} \frac{z^k}{k!} = \frac{A(z)-A(0)}{z}$
5. **Addition:** $a_0 + b_0, a_1 + b_1, \ldots, a_k + b_k, \ldots$
   $\sum_{k\geq 0} (a_k + b_k) \frac{z^k}{k!} = A(z) + B(z).$
6. **Difference:** $a_1 - a_0, a_2 - a_1, \ldots, a_{k+1} - a_k, \ldots$
   $\sum_{k\geq 0} (a_{k+1} - a_k) \frac{z^k}{k!} = A'(z) - A(z)$
7. **Binomial convolution:** $a_0 + b_0, a_1 b_0 + a_1 b_0, \ldots, \sum_{\ell=0}^{k} \binom{k}{\ell} a_\ell b_{k-\ell}, \ldots$
   $\sum_{k\geq 0} \left(\sum_{\ell=0}^{k} a_\ell b_{k-\ell} \right) \frac{z^k}{k!} = A(z)B(z).$
8. **Binomial Sum:** $a_0, a_0 + a_1 \ldots, \sum_{\ell=0}^{k} \binom{k}{\ell} a_\ell, \ldots$
   $\sum_{k\geq 0} \left(\sum_{\ell=0}^{k} a_\ell \right) \frac{z^k}{k!} = e^z A(z).$

Proof. We show now that the product of the EGFs of two sequences gives the EGF of their convolution.

$$\begin{aligned} A(z)B(z) &= \left(\sum_{k\geq 0} a_k \frac{z^k}{k!} \right) \left(\sum_{\ell\geq 0} b_\ell \frac{z^\ell}{\ell!} \right) \\ &= \sum_{k\geq 0} \left(\sum_{\ell=0}^{k} \frac{a_\ell}{\ell!} \frac{b_{k-\ell}}{(k-\ell)!} \right) z^k = \sum_{k\geq 0} \left(\sum_{\ell=0}^{k} \binom{k}{\ell} a_\ell b_{k-\ell} \right) \frac{z^k}{k!} \end{aligned}$$

The methodology for solving recurrence relations using OGF applies also to EGF.

3.2 Other Generating Functions

Other interesting generating functions are the *Dirichlet Generating Function* (DGF) and *Probability Generating Function* (PGF).

Definition 2 *For a sequence $a_0, a_1, \ldots, a_k, \ldots$, the function*

$$A(z) = \sum_{k \geq 1} \frac{a_k}{k^z}$$

is the Dirichlet Generating Function*(DGF) for the sequence.*

Definition 3 *For a random variable X that takes only non-negative integer values with $p_k = Pr\{X = k\}$, the function*

$$A(z) = \sum_{k \geq 1} p_k z^k$$

is the Probability Generating Function*(PGF) for the random variable.*

The Dirichlet generating function is a very common tool in number theory (for instance, see [1] or [4]). For example, the DGF for the sequence $1, 1, 1, \ldots$ is $\zeta(z) = \sum_{k \geq 1} \frac{1}{k^z}$, the Riemann zeta function.

3.3 Real Asymptotics

We have seen in our presentation of the analysis of Quicksort that the exact solution may contain sums or other expressions that are difficult to understand. A natural step forward is to *estimate the resulting expressions asymptotically*, that is, when the size of the input grows. That will provide quantifications that are easy to compare, and thus help in determining better algorithms on average. We did this when we used an approximation of the harmonic sum in the Quicksort study. In the following we study *real asymptotics* (later in the course we will study complex asymptotics).

Definition 4 *Given a function $f(n)$, for function $g(n)$, we write*

1. $g(n) = O(f(n))$ iff $\left|\frac{g(n)}{f(n)}\right| < k$, *for $n \to \infty$ and k a constant,*
2. $g(n) = o(f(n))$ iff $\frac{g(n)}{f(n)} \to 0$, *as $n \to \infty$, and*
3. $g(n) \sim f(n)$ iff $\frac{g(n)}{f(n)} \to 1$, *as $n \to \infty$.*

The most basic result that one can have when analyzing an algorithm is *"the average running time is $O(f(n))$"*. Indeed, this only states that the time of the algorithm is upper-bounded by a constant times $f(n)$. This hides the constant and how other terms that appear in the study of the average time are related. We avoid this kind of results when analyzing algorithms.

Average case analysis is the study of constants.

A much stronger result is $g(n) = f(n) + O(h(n))$. This says that $g(n)$ can be approximated by $f(n)$ plus a term that is within a constant factor of $h(n)$. As usual with big-Oh notation, the constant is unspecified, but we assume it is not too large. An implicit fact here is the assumption that $h(n) = o(f(n))$, otherwise we are only saying that $g(n) = O(f(n))$.

A better statement is $g(n) = f(n) + o(h(n))$. This says that $g(n)$ can be approximated by $f(n)$ plus an error term that gets smaller and smaller as $n \to \infty$. Again, we assume here an unspecified constant in the $o(h(n))$ but we assume that it is never large (even for small n), and we assume $h(n) = o(f(n))$.

We will use the notation $g(n) \sim f(n)$ to express $g(n) = f(n) + o(f(n))$. This is a weak estimation but in some cases it is the only result that we can establish.

When we have expressions involving logarithms and exponentials we want to emphasize the exponential difference of these terms. This helps in order to avoid computations that are not relevant to the solution of the problem. Example: if our quantity of interest is $2n + O(\log n)$, then it may be of no interest to improve this error term since the difference between the terms is exponential; on the other hand, $2n \log n + O(n)$ requires more study since the constant in the $O(n)$ may be large and thus our result not a very precise one. We say that the error term is *exponentially small* if it is smaller than $O\left(\frac{1}{n^m}\right)$ for any positive m. Examples of exponentially small quantities are: e^{-n}, $e^{-\log^2 n}$, etc.

3.4 Asymptotic Expansions

As we said, we prefer $f(n) = c_0 g_0(n) + O(g_1(n))$ than $f(n) = O(g_0(n))$ (we are assuming $g_1(n) = o(g_0(n))$. When $g_0(n)$ and $g_1(n)$ are close, we prefer the expression $f(n) = c_0 g_0(n)$ $+ c_1 g_1(n) + O(g_2(n))$, with $g_2(n) = o(g_1(n))$. For example, in the Quicksort algorithm we prefer $2n \ln n + 2(\gamma - 1)n + O(\log n)$ than $2n \ln n + O(n)$. For instance below is a comparison of how these values approach to the exact value $2(n+1)(H_{n+1} - 1)$ (Figure 1 is from [7], page 162).

One can of course continue getting better and better approximations. This leads to the following concept.

Definition 5 *Given a sequence of functions $\{g_k(n)\}_{k \geq 0}$ with $g_{k+1}(n) = o(g_k(n))$ for $k \geq 0$, an* asymptotic expansion of f (or asymptotic series of f) *is defined as*

$$f(n) \sim c_0 g_0(n) + c_1 g_1(n) + \ldots$$

N	$2(N+1)(H_{N+1}-1)$	$2N\ln N$	$+(2\gamma-2)N$	$+2(\ln N+\gamma)+1$
10	44.43	46.05	37.59	44.35
100	847.85	921.03	836.47	847.84
1000	12,985.91	13,815.51	12,969.94	12,985.91
10000	175,771.70	184,206.81	175,751.12	175,771.70

Figure 1: Asymptotic estimates for the comparisons in Quicksort algorithm.

This asymptotic series represents the collection of formulas:

$$\begin{aligned}
f(n) &= O\left(g_0(n)\right)\\
f(n) &= c_0g_0(n)+O\left(g_1(n)\right)\\
f(n) &= c_0g_0(n)+c_1g_1(n)+O\left(g_2(n)\right)\\
&\vdots\\
f(n) &= c_0g_0(n)+c_1g_1(n)+\ldots+c_kg_k(n)+O\left(g_{k+1}(n)\right)\\
&\vdots
\end{aligned}$$

The g_k are referred as an *asymptotic scale.*

Full asymptotic expansions exist for many functions that appear in the analysis of algorithms: harmonic numbers, factorials (Stirling's formula), Catalan numbers, etc. In other cases, we only know the first term in the asymptotic expansion, and finding more terms is an open problem. We will comment on this in the course.

In the analysis of algorithms a *finite expansion* (not an infinite asymptotic series) is all that we need. When we write

$$f(n)\sim c_0g_0(n)+c_1g_1(n)+c_2g_2(n)$$

we are implicitly assuming that the embedded error term is $o(g_2(n))$. In general we write:

$$f(n)=c_0g_0(n)+c_1g_1(n)+c_2g_2(n)+O\left(g_3(n)\right).$$

Normally, two or three terms are enough, since for large n the asymptotic scale will make later terms extremely small compared to early terms. In the above table of Quicksort the expansion $2n\ln n+2(\gamma-1)n+2\ln n+(2\gamma+1)$ gives an absolute error less than 0.1 already for $n=10$.

In general, we don't know how many terms we have to carry in order to obtain an accurate result. We take for example three or four terms, and later we redo calculations with more terms if needed.

3.5 Taylor Expansions

An important tool is *Taylor expansions*. We recall that the Taylor expansion around 0 of a function f is defined as

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \ldots$$

This shows that *a Taylor series gives rise to an asymptotic series when* $x \to 0$. When we truncate a Taylor series we obtain a finite expansion and a remainder term. In the analysis of algorithms we are interested in asymptotics when $n \to \infty$. We illustrate with some examples how we use Taylor expansion in the analysis of algorithms. First some examples of Taylor expansions.

Examples of Taylor Expansions

Exponential: $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + O\left(x^4\right).$

Logarithmic: $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + +O\left(x^4\right).$

Binomial: $(1+x)^n = 1 + nx + \binom{n}{2}x^2 + \binom{n}{3}x^3 + O\left(x^4\right).$

Geometric: $\frac{1}{1-x} = 1 + x + x^2 + x^3 + O\left(x^4\right).$

Trigonometric: $\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} + O\left(x^7\right).$

$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + O\left(x^6\right).$

Examples of Applications of Taylor Expansions

1. Find an asymptotic expansion for $\ln(n+\sqrt{n})$:

$$\ln\left(n+\sqrt{n}\right) = \ln\left(n\left(1+\frac{1}{\sqrt{n}}\right)\right) = \ln n + \ln\left(1+\frac{1}{\sqrt{n}}\right).$$

We have extracted the leading term ($\ln n$), and now we have an expression, $\ln\left(1+\frac{1}{\sqrt{n}}\right)$ which can be treated with a Taylor expansion. Setting $x = \frac{1}{\sqrt{n}}$, we notice that $x \to 0$ as $n \to \infty$. Therefore,

$$\ln\left(1+\frac{1}{\sqrt{n}}\right) = \frac{1}{\sqrt{n}} - \frac{1}{2n} + \frac{1}{3n\sqrt{n}} + O\left(\frac{1}{n^2}\right).$$

Thus we have that

$$\ln(n+\sqrt{n}) = \ln n + \frac{1}{\sqrt{n}} - \frac{1}{2n} + \frac{1}{3n\sqrt{n}} + O\left(\frac{1}{n^2}\right).$$

2. Find an asymptotic expansion for $\frac{n}{n-1}\ln\frac{n}{n-1}$:

$$\frac{n}{n-1}\ln\frac{n}{n-1} = \left(1+\frac{1}{n-1}\right)\ln\left(1+\frac{1}{n-1}\right)$$

We now apply the same trick as in the previous example. Setting $x = \frac{1}{n-1}$, we notice that $x \to 0$ as $n \to \infty$. Thus,

$$\begin{aligned}\frac{n}{n-1}\ln\frac{n}{n-1} &= \left(1+\frac{1}{n-1}\right)\left(\frac{1}{n-1}-\frac{1}{2(n-1)^2}+\frac{1}{3(n-1)^3}+O\left(\frac{1}{n^4}\right)\right)\\ &= \frac{1}{n-1}+\frac{1}{2(n-1)^2}-\frac{1}{6(n-1)^3}+\frac{1}{12(n-1)^4}+O\left(\frac{1}{n^5}\right)\end{aligned}$$

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