4.1 Manipulation Rules for Asymptotic Expansions

Below we give some rules that are useful in manipulating asymptotic expansions.

1. **Simplification:** anything smaller than the big-Oh term should not appear.
   Example. The expression $\ln n + O(1)$ is preferred to $\ln n + \gamma + O(1)$.

2. **Substitution:** change a variable to use expansions when $x \to 0$ for $n \to \infty$.
   Example. When $x = 1/n$, the expression $\ln(1+x)$ becomes
   $$\ln\left(1+\frac{1}{n}\right) = \frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} + O\left(\frac{1}{n^4}\right).$$

3. **Factoring:** take common factors and make appear expressions that can be expanded.
   Example 1. Given $\ln(n+\sqrt{n})$, rewrite it as
   $$\ln(n(1+1/\sqrt{n})) = \ln n + \ln(1+1/\sqrt{n}).$$
   Now we can expand the second term using the corresponding Taylor expansion for $x = 1/\sqrt{n}$.
   Example 2. Given $\frac{1}{n^2+n}$, rewrite it as
   $$\frac{1}{n^2}\frac{1}{1+\frac{1}{n}} = \frac{1}{n^2}\frac{1}{1-(-\frac{1}{n})}.$$
   Now we can use the Taylor expansion for $x = -1/n$.

4. **Multiplication:** use distributivity.
   Example. Consider squaring the nth harmonic number:
   $$\begin{aligned}\mathcal{H}_n^2 &= \left(\ln n + \gamma + O\left(\frac{1}{n}\right)\right)\left(\ln n + \gamma + O\left(\frac{1}{n}\right)\right)\\ &= (\ln n)^2 + 2\gamma\ln n + \gamma^2 + O\left(\frac{\ln n}{n}\right) + O\left(\frac{1}{n}\right) + O\left(\frac{1}{n^2}\right)\\ &= (\ln n)^2 + 2\gamma\ln n + \gamma^2 + O\left(\frac{\ln n}{n}\right).\end{aligned}$$
   Note that we lost in precision from $O(1/n)$ to $O((\ln n)/n)$. That is common; if we want more accuracy in the final result, we have to consider more terms in the original expansion.

5. **Exp/log trick:** when we have products, it is convenient to write them as an exponential logarithmic formula, changing to sums.
   Example. Given $(1+1/n)^n$, we can rewrite it as follows:
   $$\begin{aligned}\left(1+\frac{1}{n}\right)^n &= \exp\left(\ln\left(1+\frac{1}{n}\right)^n\right) = \exp\left(n\ln\left(1+\frac{1}{n}\right)\right)\\ &= \exp\left(n\left(\frac{1}{n}+O\left(\frac{1}{n^2}\right)\right)\right) = \exp\left(1+O\left(\frac{1}{n}\right)\right) = e^1e^{O(1/n)}.\end{aligned}$$

Using the expansion for e^x in the last expression, we can continue:

$$\begin{aligned}\left(1+\frac{1}{n}\right)^n &= e\left(1+O\left(\frac{1}{n}\right)+O\left(\left(O\left(\frac{1}{n}\right)\right)^2\right)\right)\\ &= e\left(1+O\left(\frac{1}{n}\right)\right) = e+eO\left(\frac{1}{n}\right) = e+O\left(\frac{1}{n}\right).\end{aligned}$$

Remark. The material of the remainder of this lecture is taken from Knuth [4] (Section 1.2.11). In addition, we use [5] (Sections 4.4 and 4.5); [2] (Sections 9.4 and 9.5); and [3] (Sections 4.1 and 4.2).

4.2 Bootstrapping

In the following, we describe a method for getting better and better approximations, assuming we have an implicit equation for a function of interest. Then we can treat the asymptotic information on the equation and get better approximations.

Example. Let us suppose we know

$$f(t)\cdot e^{f(t)} = t$$

when $t \to \infty$. Taking the logarithm of both sides, we get

$$f(t) = \ln t - \ln f(t). \tag{1}$$

Assume that, by some rough estimate, we determined $f(t) = O(\ln t)$. Then

$$f(t) = \ln t - \ln(O(\ln t)) = \ln t - O(\ln\ln t).$$

We put $f(t) = \ln t - O(\ln\ln t)$ inside equation (1), getting:

$$\begin{aligned}f(t) = \ln t - \ln[\ln t - O(\ln\ln t)] &= \ln t - \ln\left[\ln t\left(1+O\left(\frac{\ln\ln t}{\ln t}\right)\right)\right]\\ &= \ln t - \ln\ln t + \ln\left(1+O\left(\frac{\ln\ln t}{\ln t}\right)\right) = \ln t - \ln\ln t + O\left(\frac{\ln\ln t}{\ln t}\right),\end{aligned}$$

where the last line is obtained from the previous one using the expansion for $\ln(1+x)$. Now we can substitute the obtained expression in the last term of equation (1) to improve one more term.

4.3 Bounding the Tails of Sums

We describe another useful trick, which has several variants. Sometimes we have to compute a finite sum, but considering the infinite sum and bounding its tail gives a good solution.

Example. We have to estimate

$$n!\sum_{k=0}^{n}\frac{(-1)^k}{k!} = n!\left(\sum_{k\geq 0}\frac{(-1)^k}{k!} - \sum_{k>n}\frac{(-1)^k}{k!}\right) = n!e^{-1} - R_n,$$

where

$$R_n = n!\sum_{k>n}\frac{(-1)^k}{k!}.$$

We bound the tail as follows:

$$\begin{aligned}|R_n| &\leq n!\left(\frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \dots\right)\\ &< \frac{1}{n+1} + \frac{1}{(n+1)^2} + \cdots = \frac{1}{1-\frac{1}{n+1}} - 1 = \frac{1}{n}.\end{aligned}$$

Thus we have

$$n!e^{-1} - R_n = \frac{n!}{e} - O\left(\frac{1}{n}\right).$$

Another variant is when we *dissect the sum* of interest in pieces and treat each piece independently. In general, one has a part of the sum that contributes to the dominant result, and a tail that only contributes to the error term.

4.4 Euler-Maclaurin Formula

Now we focus on a fundamental tool of asymptotics: the *Euler-Maclaurin formula.* This tool allows the substitution of sums by integrals. The result appeared in Euler [1] and was shortly after discovered independently by Colin Maclaurin. The basic idea is to approximate the sum as follows.

Consider the graph of an arbitrary function $f(x)$ (see Figure 1). In each interval of one unit, we approximate $f(x)$ by $f(k)$, $k \leq x < k+1$. If

$$\delta_k = \max_{k\leq x<k+1}|f(x) - f(k)|$$

is the maximum error over an interval, then the total error term Δ is

$$|\Delta| = \sum_{a\leq k<b}\delta_k,$$

where a and b are the limits of the sum. Then, we have

$$\sum_{a\leq k<b} f(k) = \int_a^b f(x)dx + \Delta.$$

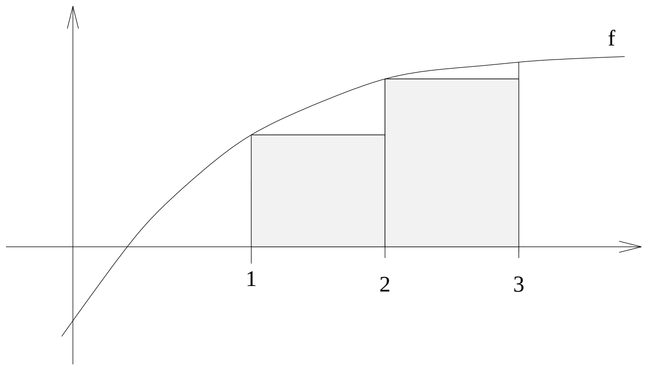

Figure 1: A graph of a function f.

When f is increasing/decreasing on the interval $[a, b]$, the error term simplifies to $\Delta \leq |f(b) - f(a)|$.

Example. Let us compute the nth harmonic number using the method just described,

$$\mathcal{H}_n = \sum_{k=1}^{n} \frac{1}{k} = \int_1^n \frac{1}{x} dx + \Delta = \ln n + \Delta,$$

where

$$\Delta \leq |f(n) - f(1)| = |1/n - 1| = (1 - 1/n).$$

If we are interested in a result like $\mathcal{H}_n = \ln n + O(1)$, this proof is sufficient. If we want a better approximation, we use the Euler-Maclaurin formula.

Theorem 1 (Euler-Maclaurin) *Let f be a function defined in the interval $[a, b]$, with a and b integers, and with derivatives $f^{(i)}$ that exist and are continuous for $1 \leq i \leq m$, m a fixed constant. Then*

$$\sum_{a \leq k < b} f(k) = \int_a^b f(x)dx + \sum_{k=1}^{m} \frac{B_k}{k!}(f^{(k-1)}(b) - f^{(k-1)}(a)) + R_m,$$

where B_k is the kth Bernoulli number*, and R_m is a remainder term satisfying*

$$|R_m| < \frac{4}{(2\pi)^m} \int_a^b |f^{(m)}(x)|dx.$$

Before we see examples of how to use this theorem, we recall the definition of Bernoulli numbers and polynomial.

Definition 1 *The coefficient B_k of the EGF $z/(e^z - 1)$ is the kth* Bernoulli number*. The* Bernoulli polynomial *is defined by the following formula:*

$$B_m(x) = \sum_{k \geq 0} \binom{m}{k} B_k x^{m-k}.$$

Some initial values of Bernoulli numbers

$$\begin{aligned} B_0 &= 1 \\ B_1 &= -\frac{1}{2} \\ B_2 &= \frac{1}{6} \\ B_3 &= B_5 = B_7 = \cdots = 0 \\ B_4 &= -\frac{1}{30} \end{aligned}$$

Properties of Bernoulli numbers/polynomials

1. $\sum_k \binom{n}{k} B_k = B_n + \delta_{n1}$, where

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We have by definition,

$$\frac{z}{e^z - 1} = \sum_{k\geq 0} B_k \frac{z^k}{k!}.$$

Multiplying both sides by $(e^z - 1)$, we get

$$z = \left(\sum_{k\geq 0} B_k \frac{z^k}{k!}\right)(e^z - 1).$$

From this we obtain

$$z + \sum_{k\geq 0} B_k \frac{z^k}{k!} = \left(\sum_{k\geq 0} B_k \frac{z^k}{k!}\right) e^z.$$

Recalling that $e^z = \sum_k \frac{z^k}{k!}$, we get the claim by using the convolution of the two EGF's on the right-hand side of the last equality, and equating the corresponding powers of z.

Q.E.D.

2. $B'_m(x) = mB_{m-1}(x)$.
Proof. We have

$$B_m(x) = \sum_k \binom{m}{k} B_k x^{m-k}.$$

Differentiating on x yields

$$\begin{aligned} B'_m(x) &= \sum_k \binom{m}{k}(m-k)B_k x^{m-k-1} \\ &= m\sum_k \binom{m-1}{k} B_k x^{m-1-k} = mB_{m-1}(x). \end{aligned}$$

Q.E.D.

3. $B_3 = B_5 = \cdots = B_{2k+1} = 0$.

4. $\sum_{0\le k<n} k^m = \frac{1}{m+1}(B_{m+1}(n) - B_{m+1}(0))$.
Proof. Consider the EGF for the sums:

$$\begin{aligned} S(z,n) &= \sum_{m\ge 0}(\sum_{0\le k<n} k^m)\frac{z^m}{m!} = \sum_{0\le k<n}\sum_{m\ge 0}\frac{(kz)^m}{m!} \\ &= \sum_{0\le k<n} e^{kz} = \frac{e^{nz}-1}{e^z-1}. \end{aligned}$$

Then,

$$\begin{aligned} S(z,n) &= \frac{z}{e^z-1}\frac{e^{nz}-1}{z} \\ &= \left(\sum_k B_k\frac{z^k}{k!}\right)\left(\frac{nz^0}{1!}+\frac{n^2z}{2!}+\frac{n^3z^2}{3!}+\dots\right). \end{aligned}$$

Performing the convolution, we obtain:

$$\begin{aligned} [z^m]S(z,n) &= (m!)\left[\frac{B_0}{0!}\frac{n^{m+1}}{(m+1)!}+\frac{B_1}{1!}\frac{n^m}{m!}+\cdots+\frac{B_m}{m!}\frac{n}{1!}\right] \\ &= \frac{1}{m+1}\left[\binom{m+1}{0}B_0n^{m+1}+\binom{m+1}{1}B_1n^m+\cdots+\binom{m+1}{m}B_mn \right.\\ &\qquad \left. +\binom{m+1}{m+1}B_{m+1}n^0-\binom{m+1}{m+1}B_{m+1}n^0\right] \\ &= \frac{1}{m+1}(B_{m+1}(n)-B_{m+1}(0)). \end{aligned}$$

Q.E.D.

Example.

(a) $m = 0$:

$$\sum_{0\le k<n} k^0 = \frac{1}{1}(B_1(n)-B_1(0)) = 1B_0n + B_1 - B_1 = n.$$

(b) $m = 1$:

$$\sum_{0 \le k < n} k = \frac{1}{2}(B_2(n) - B_2(0)) = \frac{1}{2}(B_0 n^2 + 2B_1 n + B_2 - B_2)$$
$$= \frac{1}{2}\left(1n^2 + 2\frac{-1}{2}n\right) = \frac{n^2 - n}{2} = \frac{n(n-1)}{2}.$$

(c) $m = 2$:

$$\sum_{0 \le k < n} k^2 = \frac{1}{3}(B_3(n) - B_3(0)) = \frac{1}{3}(B_0 n^3 + 3B_1 n^2 + 3B_2 n + B_3 - B_3)$$
$$= \frac{1}{3}\left(n^3 - \frac{3}{2}n^2 + \frac{3}{6}n\right) = \frac{n}{6}(2n^2 - 3n + 1) = \frac{n(n-\frac{1}{2})(n-1)}{3}$$

References

[1] L. Euler. Methodus generalis summandi progressiones. *Comentarii academiæ scientiarum imperialis Petropolitanæ*, 6:68–97, 1732. Reprinted in his Opera Omnia, series 1, volume 14, 42-72.

[2] R. Graham, D.E. Knuth, and O. Patashnik. *Concrete Mathematics*. Addison-Wesley, Reading, MA, 2 edition, 1994.

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[5] R. Sedgewick and P. Flajolet. *An Introduction to the Analysis of Algorithms*. Addison-Wesley, Reading MA, 1996.