5.1 Examples of Using the Euler-Maclaurin Formula

Remark. The material of this lecture is taken from Graham, Knuth and Patashnik [3] (Sections 9.4 and 9.5). Other sources are [4, 6, 7].

1. Harmonic Numbers

We want to estimate $H_n = \sum_{k=1}^{n} \frac{1}{k}$. Let's consider $f(x) = \frac{1}{x}$. We have $f^{(k)}(x) = (-1)^k k! x^{-k-1}$.

For this example, it turns out that the remainder term $R_m(n)$ in the Euler-Maclaurin formula is $\Theta(1)$ for any m, but we want to get an estimate that is accurate within $O(n^{-6})$. So we shall use a more precise statement of Euler's formula [3, p. 455], which says that

$$R_m(n) = \int_a^b g_m(x)dx, \text{ where } g_m(x) = (-1)^{m+1}\frac{B_m(x - \lfloor x \rfloor)}{m!} f^{(m)}(x).$$

Let $R'_m(n) = \int_n^\infty g_m(x)dx$. Then, $R_m(n) = R'_m(1) - R'_m(n)$. Fix a particular choice of m. Then, $R'_m(1)$ will just be a constant (w.r.t. n), and we shall show that $|R'_m(n)| = O(n^{-m})$ (w.r.t. n). Clearly, $|B_m(x - \lfloor x \rfloor)|$ is bounded by a constant, since the Bernoulli polynomial $B_m(y)$ is bounded for $y \in [0, 1)$. Also, $|f^{(m)}(x)|$ is a constant times x^{-m-1}, so $|g_m(x)| \leq cx^{-m-1}$ for some constant c. Thus, we know that the indefinite integral $R'_m(n)$ exists (for all n) and $|R'_m(n)| \leq \int_n^\infty cx^{-1-m}dx = \frac{c}{m}n^{-m} = O(n^{-m})$.

By the Euler-Maclaurin Formula,

$$\begin{aligned} H_{n-1} &= \int_1^n \frac{dx}{x} + \sum_{k=1}^{m} \frac{B_k}{k!}(f^{(k-1)}(n) - f^{(k-1)}(1)) + R_m(n) \\ &= \log n + \sum_{k=1}^{m} \frac{B_k}{k!}(-1)^{k-1}(k-1)!(n^{-k} - 1) + R_m(n) \\ &= \log n + \sum_{k=1}^{m} \frac{B_k}{k}(-1)^{k-1}(n^{-k} - 1) + R'_m(1) - R'_m(n). \end{aligned}$$

For example, taking $m = 6$ gives us

$$\begin{aligned} H_{n-1} &= \log n + \frac{B_1}{1}\left(\frac{1}{n} - 1\right) - \frac{B_2}{2}\left(\frac{1}{n^2} - 1\right) + 0 - \frac{B_4}{4}\left(\frac{1}{n^4} - 1\right) \\ &\quad +0 - \frac{B_6}{6}\left(\frac{1}{n^6} - 1\right) + R'_6(1) - R'_6(n) \\ &= \log n + \left(R'_6(1) + \frac{1}{2} + \frac{1}{12} - \frac{1}{120} + \frac{1}{252}\right) - \frac{1}{2n} - \frac{1}{12n^2} + \frac{1}{120n^4} + O\left(\frac{1}{n^6}\right). \end{aligned}$$

Therefore, we have

$$H_n = H_{n-1} + \frac{1}{n} = \log n + \gamma + \frac{1}{2n} - \frac{1}{12n^2} + \frac{1}{120n^4} + O\left(\frac{1}{n^6}\right),$$

where $\gamma = R_6'(1) + \frac{1}{2} + \frac{1}{12} - \frac{1}{120} + \frac{1}{252} = \lim_{n\to\infty}(H_n - \log n) = 0.5772156649\ldots$ is Euler's constant.

Using the equation derived above with $n = 10000, m = 250$, Knuth got an estimate of γ that is correct up to 1271 decimal places [5].

2. Stirling's Approximation for $n!$

We want to estimate $n!$. Using the exp-log trick, we have

$$n! = \exp(\log(n!)) = \exp\left(\sum_{k=1}^{n} \log k\right).$$

We shall estimate $\sum_{k=1}^{n-1} \log k$ using the Euler-Maclaurin Formula.

Let $f(x) = \log x$. Then, $f^{(k)}(x) = (-1)^{k-1}(k-1)!x^{-k}$ for $k \geq 1$.

We shall use the same technique (and notation) as in the previous example. This time, $|g_m(x)| \leq cx^{-m}$ for some constant c, so we can show, using the same argument, that for any fixed $m \geq 2$, $|R_m'(n)| = O(n^{1-m})$.

$$\begin{aligned}
\sum_{k=1}^{n-1} \log k &= \int_1^n \log x dx + \sum_{k=1}^{m} \frac{B_k}{k!}\left(f^{(k-1)}(n) - f^{(k-1)}(1)\right) + R_m(n) \\
&= n\log n - n + 1 - \frac{\log n}{2} + \sum_{k=2}^{m} \frac{B_k}{k!}(-1)^k(k-2)!(n^{1-k} - 1) + R_m(n) \\
&= (n - 1/2)\log n - n + 1 + \sum_{k=2}^{m} \frac{B_k}{k(k-1)}(-1)^k(n^{1-k} - 1) + R_m'(1) - R_m'(n).
\end{aligned}$$

Taking $m = 6$, we have

$$\begin{aligned}
\sum_{k=1}^{n-1} \log k &= (n - 1/2)\log n - n + 1 + \frac{1}{12}\left(\frac{1}{n} - 1\right) + 0 - \frac{1}{360}\left(\frac{1}{n^3} - 1\right) \\
&\quad + 0 + \frac{1}{1260}\left(\frac{1}{n^5} - 1\right) + R_6'(1) - R_6'(n) \\
&= (n - 1/2)\log n - n + \sigma + \frac{1}{12n} - \frac{1}{360n^3} + O\left(\frac{1}{n^5}\right),
\end{aligned}$$

where $\sigma = 1 - \frac{1}{12} + \frac{1}{360} - \frac{1}{1260} + R_6'(1)$ is Stirling's constant. So,

$$\sum_{k=1}^{n} \log k = \log n + \sum_{k=1}^{n-1} \log k = (n + 1/2)\log n - n + \sigma + \frac{1}{12n} - \frac{1}{360n^3} + O\left(\frac{1}{n^5}\right).$$

We exponentiate to get an expression for $n!$. Thus, we obtain

$$\begin{aligned} n! &= e^{(n+1/2)\log n} \cdot e^{-n} \cdot e^{\sigma} \cdot \exp\left(\frac{1}{12n} - \frac{1}{360n^3} + O\left(\frac{1}{n^5}\right)\right) \\ &= \frac{n^{n+1/2}e^{\sigma}}{e^n} \exp\left(\frac{1}{12n} - \frac{1}{360n^3} + O\left(\frac{1}{n^5}\right)\right) \\ &= e^{\sigma}\sqrt{n}\left(\frac{n}{e}\right)^n \exp\left(\frac{1}{12n} - \frac{1}{360n^3} + O\left(\frac{1}{n^5}\right)\right). \end{aligned}$$

Now we use the fact that $e^x = 1 + x + x^2/2! + x^3/3! + \cdots$ with $x = \frac{1}{12n} - \frac{1}{360n^3} + O(\frac{1}{n^5})$, and throw away any terms that are $o(n^{-5})$:

$$\begin{aligned} n! &= e^{\sigma}\sqrt{n}\left(\frac{n}{e}\right)^n \left(1 + \frac{1}{12n} - \frac{1}{360n^3} + \frac{1}{2\cdot 144n^2} - \frac{2}{2\cdot 12\cdot 360n^4} + \frac{1}{6\cdot 12^3n^3}\right. \\ &\quad \left. + \frac{1}{24\cdot 12^4n^4} + O\left(\frac{1}{n^5}\right)\right) \\ &= e^{\sigma}\sqrt{n}\left(\frac{n}{e}\right)^n \left(1 + \frac{1}{12n} + \frac{1}{288n^2} - \frac{139}{51840n^3} - \frac{571}{2488320n^4} + O\left(\frac{1}{n^5}\right)\right). \end{aligned}$$

It remains to compute e^{σ}. It turns out that this constant is $\sqrt{2\pi}$. (See Knuth, exercise 1.2.11.2.5, p. 116.)

3. Catalan Numbers

The nth Catalan number is defined by $C_n = \frac{1}{n+1}\binom{2n}{n} = \frac{(2n)!}{(n+1)(n!)^2}$. Catalan numbers are important when counting certain kinds of trees, as we shall see later in the course.

Exercise: Use Stirling's approximation to show that

$$C_n = \frac{4^n}{n\sqrt{\pi n}}\left(1 - \frac{9}{8n} + \frac{145}{128n^2} - \cdots\right).$$

5.2 Symbolic Method

Remark. The material of the symbolic method is taken from [2] that forms the first chapter of the future book by Flajolet and Sedgewick [1].

Basic Definitions

Definition 1 *A* class *of combinatorial structures is a set $\mathcal{A}$ together with a* size *function $|\cdot| : \mathcal{A} \to \mathbb{N}$. (Thus, the size of an element $\alpha \in \mathcal{A}$ is denoted $|\alpha|$.) The set of all elements of size n is denoted by A_n. We assume that A_n is a finite set for each natural number n. The number of elements in $\mathcal{A}$ of size n is denoted by a_n.*

Formally, we can define a combinatorial class by a pair $(\mathcal{A}, |\cdot|)$ *where* $\mathcal{A}$ *is a collection of objects and the function* $|\cdot| : \mathcal{A} \to \mathbb{N}$ *is such that the inverse of any integer is finite, i.e.,* $A_n = \{\alpha \in \mathcal{A} : |\alpha| = n\}$ *with* a_n *the (finite) number of elements in* A_n*.*

Definition 2 *The* counting sequence *of the class* $\mathcal{A}$ *is the sequence of natural numbers* $\{a_n\}_{n\geq 0}$*. We associate an ordinary generating function*

$$A(z) = \sum_{n\geq 0} a_n z^n$$

and an exponential generating function

$$\hat{A}(z) = \sum_{n\geq 0} a_n \frac{z^n}{n!}$$

with the counting sequence.

Since a_n represents the number of elements in $\mathcal{A}$ of size n, it is easy to see that

$$A(z) = \sum_{\alpha\in\mathcal{A}} z^{|\alpha|}$$

and

$$\hat{A}(z) = \sum_{\alpha\in\mathcal{A}} \frac{z^{|\alpha|}}{|\alpha|!}.$$

References

[1] P. Flajolet and R. Sedgewick. Analytic combinatorics. In preparation.

[2] P. Flajolet and R. Sedgewick. The average case analysis of algorithms: counting and generating functions. Technical Report 1888, INRIA, Rocquencourt, Le Chesnay, France, April 1993.

[3] R. Graham, D.E. Knuth, and O. Patashnik. *Concrete Mathematics*. Addison-Wesley, Reading, MA, 2 edition, 1994.

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[5] D.E. Knuth. Euler's constant to 1271 places. *Math. Comp.*, 16:275–281, 1962.

[6] D.E. Knuth. *The art of computer programming, vol.1: fundamental algorithms*. Addison-Wesley, Reading MA, 3 edition, 1997.

[7] R. Sedgewick and P. Flajolet. *An Introduction to the Analysis of Algorithms*. Addison-Wesley, Reading MA, 1996.