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Introduction to Mathematical Logic

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1 Introduction

Thirteen two-hour lectures and one-hour labs:

Topic	Week:	Lecture:	Lab:
Taut.	1	Tautologies $\models_p \phi$	Arithm. of $\mathcal{M} \models \phi$
	2	prop. tabl. $\vdash_p \phi$ Completeness	Manual prop. tabl. examp. tautologies
Taut. Conseq.	3	Taut. conseq., $T \models_p \phi$ Axioms in tabl., $T \vdash_p \phi$	Auto prop. tabl. examp. taut. conseq.
	4	Compactness Completeness	test
Quasi taut. Conseq.	5	Taut. conseq. from Eq $T \models_i \phi$	Taut. Conseq. from Eq
	6	Identity tabl. $T \vdash_i \phi$ red. to prop., Compl.	quasi taut. conseq. (examples)

The notes are very terse at the moment but the reader should bear in mind that they are supplemented by the labwork.

2 Language of First Order Logic

2.1 First order languages. A *first order language* $\mathcal{L}$ is given by at most countable sets of *function symbols* and *predicate symols.* Every function and predicate symbol has *arity* $n \geq 0$ which is the number of arguments the symbol expects. We require that there are effective procedures to decide whether p is an n-ary function or predicate symbol of $\mathcal{L}$.

Expressions in $\mathcal{L}$ will be *terms* and *formulas* and they will be finite sequences of symbols. Expressions will be *metamathematical* objects in contrast to mathemathical objects such as natural numbers, real numbers, topological spaces, etc. The expressions of $\mathcal{L}$ will have have no meaning by themselves. They will be defined in such a way that it will be effectively decidable whether an expression is correctly formed.

We write $\tau_1 \equiv \tau_2$ to mean that the expressions τ_1 and τ_2 are identical sequences of symbols.

In these notes we will deal exclusively with first order languages logic so from now on we will use the term *language* to denote a first order language.

2.2 Definition of terms of $\mathcal{L}$. The set of *terms* of a language $\mathcal{L}$ is the smallest set of finite sequences of symbols satisfying:

1. variables v_0, v_1, v_2, ... are terms,
2. if τ_1, ..., τ_n are terms and f is an n-ary function symbol of $\mathcal{L}$ then also $f(\tau_1, \ldots, \tau_n)$ is a term called *function application.*

Nullary function symbols are *constant* symbols and their applications $f()$, which will be abbrevaited to f, are *constants.*

We will use possibly subscripted symbols x, y, z as *syntactic* (meta) variables ranging over variables, symbols f, g as syntactic variables ranging over function symbols, and τ, ρ as syntactic variables ranging over terms.

Some binary function symbols, such as $+$, are customarily written in the *infix* form $\tau_1 + \tau_2$ as an abbreviation for $+(\tau_1, \tau_2)$.

2.3 Definition of formulas of $\mathcal{L}$. The set of *formulas* of a language $\mathcal{L}$ is the smallest set of finite sequences of symbols satisfying:

1. if τ_1 and τ_2 are terms of $\mathcal{L}$ then $\tau_1 = \tau_2$ is a formula called *identity,*
2. if τ_1, ..., τ_n are terms and P is an n-ary predicate symbol of $\mathcal{L}$ then $P(\tau_1, \ldots, \tau_n)$ is a formula called *predicate application,*
3. the symbol $\top$, called the *truth symbol,* is a formula,
4. the symbol $\bot$, called *falsehood symbol,* is a formula,
5. if ϕ is a formula so is $\neg\phi$ called *negation,*
6. if ϕ_1 and ϕ_2 are formulas so is $(\phi_1 \vee \phi_2)$, (called *disjunction*), $(\phi_1 \wedge \phi_2)$ (called *conjunction*), $(\phi_1 \rightarrow \phi_2)$ (called *implication*), and $(\phi_1 \leftrightarrow \phi_2)$ (called *equivalence*),

7. if ϕ is a formula and x a variable then so are $\forall x \phi$ (called *universal quantification*) and $\exists x \phi$ (called *existential quantification*).

Formulas formed by the rules (1) and (2) are *atomic formulas.* Formulas formed by the rules (3) though (6) are *propositional* formulas. Formulas formed by the rule (7) are *quantifier* formulas.

The formula ϕ_1 in the implication $(\phi_1 \rightarrow \phi_2)$ is called *antecedent* and ϕ_2 *consequent.* The symbols used in propositional formulas are *propositional connectives.* Nullary predicate symbols are *propositional constants* and their applications $P()$, which will be abbreviated to P, are *propositional constants.*

We use the possibly subscripted symbols ϕ, ψ as syntactic variables ranging over formulas.

In order to increase the readability of formulas we can drop the topmost pair of parentheses around a formula. All binary propositional connectives associate to the right, for instance $\phi_1 \rightarrow \phi_2 \rightarrow \phi_3$ abbreviates $\phi_1 \rightarrow (\phi_2 \rightarrow \phi_3)$. Negations and quantification have larger precedence (bind stronger) than conjunctions, which have larger precedence than disjunctions, which have larger precedence than implications and equivalences. We often abbreviate $\neg \tau_1 = \tau_2$ to $\tau_1 \neq \tau_2$.

2.4 Metamathematical implication and equivalence. In order to shorten the metamathematical discursion in English we will use the symbol $\Rightarrow$ in context $\cdots \Rightarrow \cdots$ as abbreviation for *if* $\cdots$ *then* $\cdots$ and the symbol $\Leftrightarrow$ in context $\cdots \Leftrightarrow \cdots$ as abbreviation for $\cdots$ *if and only if* $\cdots$. Sometimes we will write the last also as $\cdots$ *iff* $\cdots$. We use the metamathematical symbols $\Rightarrow$ and $\Leftrightarrow$ in order to distinguish them from the logical symbols $\rightarrow$ and $\leftrightarrow$ wchich are from the *object*, i.e. first order, language.

3 Tautologies

In this section we investigate formulas which are always true only on the strength of their propositional connectives.

Semantics

3.1 Propositional atoms. Atomic and quantified formulas of $\mathcal{L}$ are called the *propositional atoms* of $\mathcal{L}$. The metamathematical function $FPA(\alpha)$ is defined to yield the set of *free* propositional atoms of α, i.e. propositional atoms of α outside of quantifiers. The function is defined for formulas $\alpha \equiv \phi$ and sets of formulas $\alpha \equiv T$ to satisfy:

$$
\begin{aligned}
FPA(\tau_1 = \tau_2) &= \{\tau_1 = \tau_2\} \\
FPA(P(\tau_1, \ldots, \tau_n)) &= \{P(\tau_1, \ldots, \tau_n)\} \\
FPA(\forall x \phi) &= \{\forall x \phi\} \\
FPA(\exists x \phi) &= \{\exists x \phi\} \\
FPA(\top) &= \emptyset \\
FPA(\bot) &= \emptyset \\
FPA(\neg \phi) &= FPA(\phi) \\
FPA(\phi_1 \vee \phi_2) &= FPA(\phi_1) \cup FPA(\phi_2) \\
FPA(\phi_1 \wedge \phi_2) &= FPA(\phi_1) \cup FPA(\phi_2) \\
FPA(\phi_1 \rightarrow \phi_2) &= FPA(\phi_1) \cup FPA(\phi_2) \\
FPA(\phi_1 \leftrightarrow \phi_2) &= FPA(\phi_1) \cup FPA(\phi_2) \\
FPA(T) &= \bigcup \{FPA(\phi) \mid \phi \in T\} \ .
\end{aligned}
$$

3.2 Propositional interpretations. A *propositional interpretation* $\mathcal{M}$ for $\mathcal{L}$ is a subset of propositional atoms of $\mathcal{L}$. $\mathcal{M}$ is *finite* or *infinite* if the set $\mathcal{M}$ is finite or infinite. The intention is that a propositional atom ϕ is true in the propositional interpretation $\mathcal{M}$ iff $\phi \in \mathcal{M}$.

We denote by $\mathcal{M}^T$ the *restriction of* $\mathcal{M}$ *to the free propositional atoms of the set* T: $\mathcal{M}^T = \mathcal{M} \cap FPA(T)$.

3.3 Satisfaction relation for propositional interpretations. For a given propositional interpretation $\mathcal{M}$ for $\mathcal{L}$ and a formula ϕ we define the unary relation $\mathcal{M}$ *satisfies* ϕ, written as $\mathcal{M} \models \phi$, as follows where ϕ, ϕ_1 and ϕ_2 are arbitrary formulas and ψ are propositional atoms of $\mathcal{L}$:

$$
\begin{aligned}
\mathcal{M} \models \psi &\Leftrightarrow \psi \in \mathcal{M} \\
\mathcal{M} &\models \top \\
\mathcal{M} &\not\models \bot \\
\mathcal{M} \models \neg \phi &\Leftrightarrow \mathcal{M} \not\models \phi \\
\mathcal{M} \models \phi_1 \vee \phi_2 &\Leftrightarrow \mathcal{M} \models \phi_1 \text{ or } \mathcal{M} \models \phi_2 \\
\mathcal{M} \models \phi_1 \wedge \phi_2 &\Leftrightarrow \mathcal{M} \models \phi_1 \text{ and } \mathcal{M} \models \phi_2 \\
\mathcal{M} \models \phi_1 \rightarrow \phi_2 &\Leftrightarrow \mathcal{M} \not\models \phi_1 \text{ or } \mathcal{M} \models \phi_2 \\
\mathcal{M} \models \phi_1 \leftrightarrow \phi_2 &\Leftrightarrow \mathcal{M} \models \phi_1 \rightarrow \phi_2 \text{ and } \mathcal{M} \models \phi_2 \rightarrow \phi_1 \ .
\end{aligned}
$$

Note that the propositional atoms obtain meaning directly from $\mathcal{M}$ whereas the meaning of other kinds of propositional formulas is uniquely determined by the meaning of its subformulas.

We write $\mathcal{M} \models T$ as an abbreviation for $\mathcal{M} \models \phi$ *for all* $\phi \in T$ and say that T *is satisfied in* $\mathcal{M}$.

Two (propositional) interpretations $\mathcal{M}$ and $\mathcal{N}$ for $\mathcal{L}$ are T*-equivalent*, in writing $\mathcal{M} \equiv_T \mathcal{N}$, if

$$\mathcal{M} \models T \Leftrightarrow \mathcal{N} \models T \ .$$

$\mathcal{M}$ and $\mathcal{N}$ are *equivalent*, in writing $\mathcal{M} \equiv \mathcal{N}$, if they are $\{\phi \mid \phi \in \mathcal{L}\}$-equivalent, i.e. $\mathcal{M} \models \phi \Leftrightarrow \mathcal{N} \models \phi$ holds for all formulas ϕ.

3.4 Sequents. Fix a language $\mathcal{L}$. For a finite sequence of formulas Λ: ϕ_1, ..., ϕ_n we write $\bigwedge\Lambda$ as an abbreviation for the conjunction $\phi_1 \wedge \cdots \wedge \phi_n$ if $n \geq 1$ and for $\top$ if $n = 0$. For any propositional interpretation $\mathcal{M}$ we clearly have $\mathcal{M} \models \bigwedge\Lambda$ iff $\mathcal{M} \models \phi$ for all $\phi \in \Lambda$. Note that this holds also for $\Lambda \equiv \emptyset$ because $\bigwedge\emptyset \equiv \top$ for which $\mathcal{M} \models \top$ holds just as $\mathcal{M} \models \phi$ vacuously holds for all $\phi \in \emptyset$.

For the same sequence Λ we write $\bigvee\Lambda$ as an abbreviation for the disjunction $\phi_1 \vee \cdots \vee \phi_n$ if $n \geq 1$ and for $\bot$ if $n = 0$. For any propositional interpretation $\mathcal{M}$ we clearly have $\mathcal{M} \models \bigvee\Lambda$ iff $\mathcal{M} \models \phi$ for some $\phi \in \Lambda$.

For two finite sequences of formulas Λ and Γ we call the formula $\bigwedge\Lambda \to \bigvee\Gamma$ a *sequent*. For any propositional interpretation $\mathcal{M}$ we clearly have $\mathcal{M} \models \bigwedge\Lambda \to \bigvee\Gamma$ iff $\mathcal{M} \not\models \phi$ for some $\phi \in \Lambda$ or $\mathcal{M} \models \phi$ for some $\phi \in \Gamma$. Thus $\mathcal{M} \not\models \bigwedge\Lambda \to \bigvee\Gamma$ iff $\mathcal{M} \models \phi$ for all $\phi \in \Lambda$ and $\mathcal{M} \not\models \phi$ for all $\phi \in \Gamma$. We have $\mathcal{M} \not\models \bigwedge\emptyset \to \bigvee\emptyset$ because this stands for $\mathcal{M} \not\models \top \to \bot$.

3.5 Tautologies. A formula ϕ of $\mathcal{L}$ is a *tautology*, in symbols $\models_p \phi$, if $\mathcal{M} \models \phi$ holds for all propositional interpretations $\mathcal{M}$ for $\mathcal{L}$.

We say that two formulas ϕ_1 and ϕ_2 are *propositionally equivalent* if we have $\mathcal{M} \models \phi_1 \Leftrightarrow \mathcal{M} \models \phi_2$ for all propositional interpretations $\mathcal{M}$. But this means that $\mathcal{M} \models \phi_1 \leftrightarrow \phi_2$ holds for all propositional interpretations $\mathcal{M}$ and so $\phi_1 \leftrightarrow \phi_2$ is a tautology. Note that we then also have $\models_p \phi_1$ iff $\models_p \phi_2$.

Because $\mathcal{M} \models \phi_1 \wedge \phi_2$ holds iff $\mathcal{M} \models \phi_1$ and $\mathcal{M} \models \phi_2$ hold we have $\models_p \phi_1 \wedge \phi_2$ iff $\models_p \phi_1$ and $\models_p \phi_2$.

3.6 Equivalence lemma.

$$\mathcal{M}^T = \mathcal{N}^T \Rightarrow \mathcal{M} \equiv_T \mathcal{N} \ .$$

Proof. Assume $\mathcal{M}^T = \mathcal{N}^T$ and for any $\phi \in T$ prove $\mathcal{M} \models \phi \Leftrightarrow \mathcal{N} \models \phi$ by a straightforward induction on the construction of ϕ because for every $\phi \in FPA(\phi)$ we have $\phi \in \mathcal{M} \Leftrightarrow \phi \in \mathcal{N}$. □

3.7 Decidability of tautologies. Fix a language $\mathcal{L}$. We call a set S of formulas of $\mathcal{L}$ *decidable* if we can effectively decide for every formula ϕ whether or not $\phi \in S$ holds. Finite sets S are clearly decidable.

For every decidable propositional interpretation $\mathcal{M}$ and for every formula ϕ of $\mathcal{L}$ we can effectively decide whether $\mathcal{M} \models \phi$ holds by simply using the properties Par. 3.3 of propositional truth.

We can effectively decide whether a formula ϕ is a tautology because the set $FPA\phi$ of all of its free propositional atoms contains finitely many, say n, atoms. We claim that $\models_p \phi$ holds iff $\mathcal{M} \models_p \phi$ holds for all 2^n subsets $\mathcal{M}$ of

$FPA(\phi)$ and the last is clearly effectively decidable. The claim holds because if ϕ is a tautology then it is true in all subsets of $FPA(\phi)$. On the other hand, if ϕ is true in all subsets of $FPA(\phi)$ then if $\mathcal{M}$ is any propositional interpretation for $\mathcal{L}$ then so is $\mathcal{M}^\phi \subseteq FPA(\phi)$ and we have $\mathcal{M}^\phi \models \phi$ and hence $\mathcal{M} \models \phi$ by Lemma 3.6. Thus ϕ is a tautology.

The just described method of deciding whether ϕ is a tautology is called the *truth table method* because every propositional interpretation $\mathcal{M} \subseteq FPA(\phi)$ can be viewed as a row in a table containing t or f according to whether $\mathcal{M} \models \phi$ holds or not.

Syntax

We now introduce a method of testing for tautologies by propositional *tableau* proofs. Tableaux were first used by Beth and refined by R. Smullyan. We use the signed tableaux of Smullyan but in a positive way which proves goals rather than refuting them. We will show that tableau proofs prove exactly the tautologies. We introduce the tableaux because they can be extended to deal with identity and quantifiers whereas the truth table method cannot, at least not directly.

3.8 Signed formulas. A formula ϕ of a language $\mathcal{L}$ is *signed* if it is written in one of two forms: ϕ or $\phi*$. The signed formula of the first form is called *assumption* and of the second form *goal*. We will use ϕ^+ as a syntactic variable ranging over signed formulas. We will denote by Δ finite nonempty sequences of signed formulas. Every sequence Δ of signed formulas can be associated with a sequent $\bigwedge \Lambda \to \bigvee \Gamma$ where the sequence Λ contains exactly the assumptions in Δ and the sequence Γ contains exactly the goals in Δ (with the signs $*$ deleted).

3.9 Propositional tableaux. Fix a language $\mathcal{L}$. Propositional tableaux for $\mathcal{L}$ are finitely branching trees built by to two kinds of expansion rules. A *unary tableau expansion* is of the form

$$\frac{\Delta_1}{\phi_1^+} \tag{1}$$

where Δ_1 is a possibly empty sequence of signed formulas and ϕ_1^+ a signed formula. A *binary tableau expansion rule* is of the form

$$\frac{\Delta_1}{\phi_1^+ \mid \phi_2^+} \tag{2}$$

where also ϕ_2^+ is a signed formula. The signed formulas of Δ are called *premises* and the signed formulas ϕ_1^+ and ϕ_2^+ *conclusions* of the expansion rules.

Let Δ be a non-empty sequence of signed formulas and $\bigwedge \Lambda \to \bigvee \Gamma$ a sequent associated with Δ. A binary tree π with signed formulas as labels is a *tableau for* Δ:

$$\frac{\Delta}{\cdots\pi\cdots}$$

if the tree π is either empty or it is obtained by an expansion rule. In the graphical representation we place the tableau π under the sequence of signed formulas Δ. The tableau is separated from the sequence by a solid line.

If the tableau π for Δ is empty then it has the form

$$\frac{\Delta}{\circ}$$

If the tableau π for Δ is obtained by a unary rule (1) then it has the form

$$\frac{\Delta}{\begin{array}{c}\phi_1^+\\ \cdots\pi_1\cdots\end{array}} \quad \text{where } \Delta_1 \subseteq \Delta \text{ and } \quad \frac{\begin{array}{c}\Delta\\ \phi_1^+\end{array}}{\cdots\pi_1\cdots}$$

Note that π_1 is a tableau for the sequence Δ, ϕ_1^+. The unary propositional rules will be such that

$$\models_p (\bigwedge \Lambda \to \bigvee \Gamma) \leftrightarrow (\bigwedge \Lambda_1 \to \bigvee \Gamma_1) \tag{3}$$

where the sequent on the right is associated with Δ, ϕ_1^+.

If the tableau π for Δ is obtained by a binary rule (2) then it has the form

$$\frac{\Delta}{\begin{array}{ccc}\phi_1^+ & \diagup\!\!\diagdown & \phi_2^+\\ \cdots\pi_1\cdots & & \cdots\pi_2\cdots\end{array}} \quad \text{where } \Delta_1 \subseteq \Delta, \quad \frac{\begin{array}{c}\Delta\\ \phi_1^+\end{array}}{\cdots\pi_1\cdots} \quad \text{, and } \quad \frac{\begin{array}{c}\Delta\\ \phi_2^+\end{array}}{\cdots\pi_2\cdots}$$

Note that π_1 is a tableau for the sequence Δ, ϕ_1 and π_2 a tableau for the sequence Δ, ϕ_2. The binary rules will be such that

$$\models_p (\bigwedge \Lambda \to \bigvee \Gamma) \leftrightarrow (\bigwedge \Lambda_1 \to \bigvee \Gamma_1) \land (\bigwedge \Lambda_2 \to \bigvee \Gamma_2) \tag{4}$$

where the two sequents on the right are associated with Δ, ϕ_1^+ and Δ, ϕ_2^+ respectively.

3.10 Propositional tableau expansion rules. Fix a language $\mathcal{L}$. The following unary propositional tableau expansion rules are called *flatten rules*:

$$\frac{\phi_1 \land \phi_2}{\phi_1}\,(\land_1) \quad \frac{\phi_1 \land \phi_2}{\phi_2}\,(\land_2) \quad \frac{\phi_1 \leftrightarrow \phi_2}{\phi_1 \to \phi_2}\,(\leftrightarrow_1) \quad \frac{\phi_1 \leftrightarrow \phi_2}{\phi_2 \to \phi_1}\,(\leftrightarrow_2)$$

$$\frac{\phi_1 \to \phi_2 *}{\phi_1}\,(\to_1 *) \quad \frac{\phi_1 \to \phi_2 *}{\phi_2 *}\,(\to_2 *) \quad \frac{\phi_1 \lor \phi_2 *}{\phi_1 *}\,(\lor_1 *) \quad \frac{\phi_1 \lor \phi_2 *}{\phi_2 *}\,(\lor_2 *)\ .$$

The following binary propositional tableau expansion rules are called *split rules*:

$$\frac{\phi_1 \vee \phi_2}{\phi_1 \mid \phi_2}\,(\vee) \quad \frac{\phi_1 \to \phi_2}{\phi_2 \mid \phi_1 *}\,(\to) \quad \frac{\phi_1 \wedge \phi_2 *}{\phi_1 * \mid \phi_2 *}\,(\wedge *) \quad \frac{\phi_1 \leftrightarrow \phi_2 *}{\phi_1 \to \phi_2 * \mid \phi_2 \to \phi_1 *}\,(\leftrightarrow *)\ .$$

The following unary propositional tableau expansion rules are called *inversion rules*:

$$\frac{\neg\phi}{\phi *}\,(\neg) \quad \frac{\neg\phi *}{\phi}\,(\neg *)\ .$$

Note that we have for each propositional connective two rules, one when the connective is the premise formula as an assumption and one as a goal.

We will now convince ourselves by the truth table method that, for instance, the rule $\to$ satisfies 3.9(4) which has the form:

$$\models_p ((\phi_1 \to \phi_2) \wedge \bigwedge \Lambda \to \bigvee \Gamma) \leftrightarrow (\phi_2 \wedge \bigwedge \Lambda \to \bigvee \Gamma) \wedge (\bigwedge \Lambda \to \phi_1 \vee \bigvee \Gamma)\ .$$

Indeed, for every propositional interpretation $\mathcal{M}$ we have

$$\mathcal{M} \models (\phi_1 \to \phi_2) \wedge \bigwedge \Lambda \to \bigvee \Gamma$$

iff $\mathcal{M} \not\models \phi_1 \to \phi_2$ or $\mathcal{M} \models \bigwedge \Lambda \to \bigvee \Gamma$ iff ($\mathcal{M} \models \phi_1$ and $\mathcal{M} \not\models \phi_2$) or $\mathcal{M} \models \bigwedge \Lambda \to \bigvee \Gamma$ iff $\mathcal{M} \models \bigwedge \Lambda \to \phi_1 \vee \bigvee \Gamma$ and $\mathcal{M} \models \phi_2 \wedge \bigwedge \Lambda \to \bigvee \Gamma$ iff

$$\mathcal{M} \models (\phi_2 \wedge \bigwedge \Lambda \to \bigvee \Gamma) \wedge (\bigwedge \Lambda \to \phi_1 \vee \bigvee \Gamma)\ .$$

3.11 Proofs with propositional tableaux. Fix a language $\mathcal{L}$. Let Δ be a non-empty finite sequence of signed formulas of $\mathcal{L}$, and π a propositional tableau for Δ.

A *branch* of the tableau π is a sequence of signed formulas Δ, Δ_1 where Δ_1 is read off some branch of the tree π. We say that the branch is *closed on* ϕ if both ϕ and $\phi *$ are in the branch Δ, Δ_1. The branch is *closed* if Δ, Δ_1 contains either the goal $\top *$, or the assumption $\bot$, or the branch is closed on some ϕ different from $\top$ and $\bot$. The tableau π is *closed* if all of its branches are closed.

We write $\pi\colon \vdash_p [\Delta]$ when π is a closed propositional tableau for Δ. We write $\vdash_p [\Delta]$ if there is a propositional tableau π such that $\pi\colon \vdash_p [\Delta]$. By a simple induction proof on the structure of π we can show:

$$\pi : \ \vdash_p [\Delta] \text{ and } \Delta \subseteq \Delta_1 \Rightarrow \pi : \ \vdash_p [\Delta_1]$$

where by $\Delta \subseteq \Delta_1$ we mean $\phi^+ \in \Delta \Rightarrow \phi^+ \in \Delta_1$ *for all* ϕ^+.

For a formula ϕ we say that the tableau π *proves* ϕ, in symbols $\pi\colon \vdash_p \phi$, if $\pi\colon \vdash_p [\phi *]$ holds. The same conventions as for sequences Δ apply also to formulas ϕ. In particular, we say that ϕ *is provable* if $\vdash_p \phi$ holds. For a set of of formulas S we write $\vdash_p S$ if $\vdash_p \phi$ holds for all $\phi \in S$.

3.12 Lemma (Soundness of propositional tableaux). *If* $\bigwedge\Lambda \to \bigvee\Gamma$ *is associated to* Δ *then*

$$\pi\colon \vdash_p [\Delta] \Rightarrow\ \models_p \bigwedge\Lambda \to \bigvee\Gamma\ .$$

Proof. By induction on the structure of π. So assume that the tableau π for Δ is closed:

$$\frac{\Delta}{\cdots\pi\cdots}$$

and perform a case analysis on π. If π is empty then either $\top* \in \Delta$, i.e. $\top \in \Gamma$, or $\bot \in \Delta$, i.e. $\bot \in \Lambda$, or else $\psi, \psi* \in \Delta$, i.e. $\psi \in \Lambda$ and $\psi \in \Gamma$ for some propositional atom ψ. For any propositional interpretation $\mathcal{M}$ we then have $\mathcal{M} \models \bigwedge\Lambda \to \bigvee\Gamma$ and so the sequent is a tautology.

If the first expansion in π is by a unary rule 3.9(1) such that $\Delta_1 \subseteq \Delta$ then we have

$$\frac{\Delta}{\begin{array}{c}\phi_1^+\\ \cdots\pi_1\cdots\end{array}} \quad \text{i.e.} \quad \frac{\begin{array}{c}\Delta\\ \phi_1^+\end{array}}{\cdots\pi_1\cdots}$$

for some tableau π_1 such that $\pi_1\colon \vdash_p [\Delta, \phi_1^+]$. Let $\bigwedge\Lambda_1 \to \bigvee\Gamma_1$ be a sequent associated with Δ, ϕ_1^+. We get $\models_p \bigwedge\Lambda_1 \to \bigvee\Gamma_1$ by IH and so $\models_p \bigwedge\Lambda \to \bigvee\Gamma$ by 3.9(3).

If the first expansion in π is by a binary rule 3.9(2) such that $\Delta_1 \subseteq \Delta$ then we have

$$\frac{\Delta}{\begin{array}{cc}\phi_1^+ & \phi_2^+\\ \cdots\pi_1\cdots & \cdots\pi_2\cdots\end{array}} \quad \text{i.e.} \quad \frac{\begin{array}{c}\Delta\\ \phi_1^+\end{array}}{\cdots\pi_1\cdots} \quad \text{and} \quad \frac{\begin{array}{c}\Delta\\ \phi_2^+\end{array}}{\cdots\pi_2\cdots}$$

for some tableaux π_1 and π_2 such that $\pi_1\colon \vdash_p [\Delta, \phi_1^+]$ and $\pi_2\colon \vdash_p [\Delta, \phi_2^+]$. Let $\bigwedge\Lambda_1 \to \bigvee\Gamma_1$ and $\bigwedge\Lambda_2 \to \bigvee\Gamma_2$ be sequents associated with Δ, ϕ_1^+ and Δ, ϕ_2^+ respectively. We get $\models_p \bigwedge\Lambda_1 \to \bigvee\Gamma_1$ and $\models_p \bigwedge\Lambda_2 \to \bigvee\Gamma_2$ by two IH's. Hence $\models_p \bigwedge\Lambda \to \bigvee\Gamma$ by 3.9(4). □

3.13 Lemma (Completeness of propositional tableaux). *If* $\bigwedge\Lambda \to \bigvee\Gamma$ *is associated with* Δ *then*

$$\models_p \bigwedge\Lambda \to \bigvee\Gamma \Rightarrow\ \vdash_p [\Delta]\ .$$

Proof. Assume $\models_p \bigwedge\Lambda \to \bigvee\Gamma$ and prove $\vdash_p [\Delta]$ by induction on the total number n of propositional connectives in the formulas of Δ (not counting those within quantifiers).

If $n = 0$ then Δ consists at most of propositional atoms $\bot$ and $\top$. If $\bot \in \Lambda$ or $\top \in \Gamma$ then the branch Δ is closed. If neither of the two cases applies and the sequences Λ and Γ have no propositional atom in common then we have a contradiction because $\mathcal{M} \not\models \bigwedge\Lambda \to \bigvee\Gamma$ for $\mathcal{M} = \{\phi \mid \phi \in \Lambda\}$. Thus Δ must be closed on some ϕ. Hence, in both cases it suffices to take π empty.

If $n > 0$ then select a signed propositional formula ϕ^+ of Δ which is not $\top$ or $\bot$ and denote by Δ_1 the sequence obtained from Δ by omitting the selected formula from it. Denote by Λ_1 the sequence obtained from Λ by deleting ϕ if the selected signed formula is an assumption and the sequence Λ otherwise and denote by Γ_1 the sequence obtained from Γ by deleting ϕ if the selected formula is a goal and the sequence Γ otherwise. Thus if the selected formula is a goal then $\bigwedge\Lambda_1 \to \phi \lor \bigvee\Gamma_1$ is a tautology and if the selected formula is an assumption then $\phi \land \bigwedge\Lambda_1 \to \bigvee\Gamma_1$ is a tautology. We wish to prove $\vdash_p [\Delta]$ by the case analysis of the signed formula ϕ^+.

If $\phi^+ \equiv \neg\phi_1 * \in \Delta$ then, since $\models_p \bigwedge\Lambda_1 \to \neg\phi_1 \lor \bigvee\Gamma_1$, we also have $\models_p \phi_1 \land \bigwedge\Lambda_1 \to \bigvee\Gamma_1$. The associated sequence Δ_1, ϕ_1 has $n-1$ connectives and so $\pi_1\colon \vdash_p [\Delta_1, \phi_1]$ for some π_1 by IH. This is shown on the left and the constructed tableau is shown on the right:

$$\begin{array}{c} \Delta_1 \\ \phi_1 \\ \hline \cdots\pi_1\cdots \end{array} \quad\Rightarrow\quad \begin{array}{cc} \Delta & \\ \hline \phi_1 & (\neg *) \\ \cdots\pi_1\cdots & \end{array}$$

The constructed tableau is closed and so $\vdash_p [\Delta]$.

If $\phi^+ \equiv \phi_1 \lor \phi_2 * \in \Delta$ then, since $\models_p \bigwedge\Lambda_1 \to (\phi_1 \lor \phi_2) \lor \bigvee\Gamma_1$, we also have

$$\models_p \bigwedge\Lambda_1 \to \phi_1 \lor \phi_2 \lor \bigvee\Gamma_1 \ .$$

The associated sequence $\Delta_1, \phi_1 *, \phi_2 *$ has $n-1$ propositional connectives and so $\pi_1\colon \vdash_p [\Delta_1, \phi_1 *, \phi_2 *]$ for some π_1 by IH. This is shown in the following on the left and the constructed tableau is shown on the right:

$$\begin{array}{c} \Delta_1 \\ \phi_1 * \\ \phi_2 * \\ \hline \cdots\pi_1\cdots \end{array} \quad\Rightarrow\quad \begin{array}{cc} \Delta & \\ \hline \phi_1 * & (\lor_1 *) \\ \phi_2 * & (\lor_2 *) \\ \cdots\pi_1\cdots & \end{array}$$

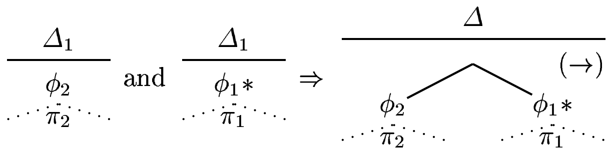
The constructed tableau is closed and so $\vdash_p [\Delta]$.

If $\phi^+ \equiv \phi_1 \to \phi_2 \in \Delta$ then, since $\models_p (\phi_1 \to \phi_2) \land \bigwedge\Lambda_1 \to \bigvee\Gamma_1$, we also have

$$\models_p \phi_2 \land \bigwedge\Lambda_1 \to \bigvee\Gamma_1 \text{ and } \models_p \bigwedge\Lambda_1 \to \phi_1 \lor \bigvee\Gamma_1 \ .$$

The sequences Δ_1, ϕ_2 and $\Delta_1, \phi_1 *$ have $n-1$ propositional connectives each and so $\pi_2\colon \vdash_p [\Delta_1, \phi_2]$ and $\pi_1\colon \vdash_p [\Delta_1, \phi_1 *]$ for some π_2 and π_1 by two IH's.

The two tableaux are shown on the left and the constructed tableau is shown on the right:



The constructed tableau is closed and so $\vdash_p [\Delta]$. The remaining cases are similar. □

3.14 Theorem (Soundness and completeness of propositional tableaux).

$$\vdash_p \phi \Leftrightarrow \models_p \phi \ .$$

Proof. We have $\vdash_p \phi$ iff by definition $\vdash_p [\phi*]$ iff by Lemmas 3.12 and 3.13 $\models_p \top \to \phi$ iff $\models_p \phi$. □

3.15 Falsification of open branches in propositional tableaux. A branch Δ of a propositional tableau π for $\phi*$ is *propositionally complete* if with every expansion rule with a premise in Δ the branch contains also at least one of its conclusions.

If a tableau for $\phi*$ does not close then it contains an open branch Δ and this can be clearly propositionally completed by finitely many expansions because the conclusions of expansion rules are formulas with a lesser number of propositional connectives than the premises.

For every propositionally complete and not closed branch Δ of a tableau for ϕ we can construct a propositional interpretation $\mathcal{M}$ by collecting all propositional atoms in assumptions:

$$\mathcal{M} = \{\psi \mid \psi \in \Delta \text{ and } \psi \text{ is a propositional atom}\} \ .$$

We prove by induction on the number of connectives in ψ that the following holds:

$$(\psi \in \Delta \Rightarrow \mathcal{M} \models \psi) \text{ and } (\psi* \in \Delta \Rightarrow \mathcal{M} \not\models \psi) \ .$$

If ψ is a propositional atom then the claim holds directly from the definition of $\mathcal{M}$. If $\psi \equiv \neg\psi_1$ then if $\neg\psi_1 \in \Delta$ we have $\psi_1* \in \Delta$ because the branch is propositionally complete and so $\mathcal{M} \not\models \psi_1$ by IH, i.e. $\mathcal{M} \models \neg\psi_1$. The case $\neg\psi_1* \in \Delta$ is similar.

If $\psi \equiv \psi_1 \wedge \psi_2$ then if $\psi_1 \wedge \psi_2 \in \Delta$ we have $\psi_1, \psi_2 \in \Delta$ because of saturation and $\mathcal{M} \models \psi_1$, $\mathcal{M} \models \psi_2$ by two IH's. Hence $\mathcal{M} \models \psi_1 \wedge \psi_2$. If $\psi_1 \wedge \psi_2* \in \Delta$ then one of the subformulas is a goal in Δ, say $\psi_1* \in \Delta$. Thus $\mathcal{M} \not\models \psi_1$ by IH and hence $\mathcal{M} \not\models \psi_1 \wedge \psi_2$. The remaining cases for ψ are similar.

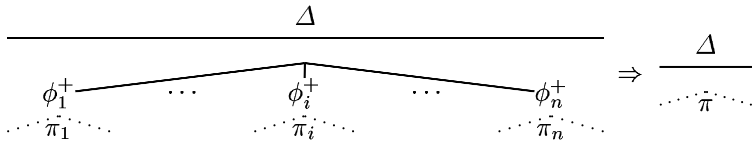
The goal $\phi*$ cannot be a tautology because we have $\mathcal{M} \not\models \phi$ by the just proved property.

4 Admissible Expansion Rules

4.1 Admissible expansion rules. We can sometimes considerably shorten a tableau proof by the use of *admissible* expansion rules. Admissible rules can be derived from the basic expansion rules (in the propositional case the rules are given in Par. 3.10) and so they do not add any strength to the proof system. The n-ary expansion rule ($n \geq 1$)

$$\frac{\Delta_1}{\phi_1^+ \mid \cdots \mid \phi_i^+ \mid \cdots \mid \phi_n^+}$$

is *admissible* if for every sequence of signed formulas Δ s.t. $\Delta_1 \subseteq \Delta$ and for all tableaux $\pi_1, \ldots, \pi_i, \ldots, \pi_n$ such that $\pi_i\colon \vdash_p [\Delta, \phi_i^+]$ we have $\pi\colon \vdash_p [\Delta]$ for some π. The situation can be visualized as follows:



where an application of the admissible rule and the subsequent closure by tableaux π_i on the left can be replaced by the basic rules in π.

4.2 Theorem (Generalized flatten rules). *Following* generalized flatten *rules are admissible in propositional tableaux for any* $n \geq 2$ *and* $1 \leq i \leq n$*:*

$$\frac{\phi_1 \wedge \cdots \wedge \phi_i \wedge \cdots \wedge \phi_n}{\phi_i}\ (G\wedge_i)$$

$$\frac{\phi_1 \vee \cdots \vee \phi_i \vee \cdots \vee \phi_n *}{\phi_i *}\ (G\vee_i *)\ .$$

Proof. We prove the admissibility of the rule $(G\wedge_i)$ by induction on i; the admissibility of $(G\vee_i *)$ is proved similarly. In the base case when $i = 1$ there is nothing to prove as $(G\wedge_1)$ is the basic flatten rule $(\wedge_1)$. For $2 \leq i + 1 \leq n$ we consider an expansion by the rule:

$$\begin{array}{c} \vdots \\ \phi_1 \wedge \phi_2 \wedge \cdots \wedge \phi_n \\ \vdots \\ \hline \phi_{i+1} \qquad (G\wedge_{i+1}) \\ \pi \end{array}$$

If $n = 2$ then this is the basic $(\wedge_2)$ rule and we are done. If $n > 2$ then we replace the rule $(G\wedge_{i+1})$ by the basic rule $(\wedge_2)$ followed by the $(G\wedge_i)$ rule which is admissible by IH. The new tableau is as follows:

$$\begin{array}{ll}
 & \vdots \\
 & \phi_1 \wedge \phi_2 \wedge \cdots \wedge \phi_n \\
 & \vdots \\
\hline
\phi_2 \wedge \cdots \wedge \phi_n & (\wedge_2) \\
\phi_{i+1} & (G\wedge_i), \mathrm{IH} \\
\pi &
\end{array}$$

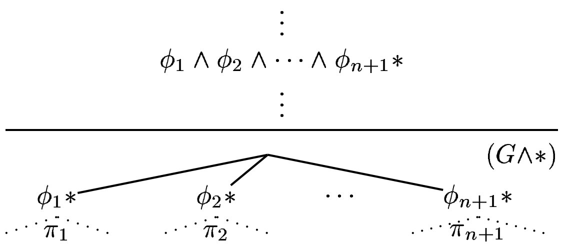
□

4.3 Theorem (Generalized split rules). *The following generalization of the rule (→) is admissible in propositional tableaux for any $n \geq 1$:*

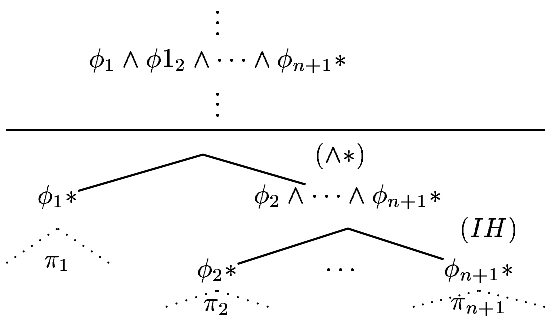
$$\frac{\phi_1 \vee \cdots \vee \phi_n}{\phi_1 \mid \cdots \mid \phi_n} \quad (G\vee)$$

$$\frac{\phi_1 \wedge \cdots \wedge \phi_n *}{\phi_1 * \mid \cdots \mid \phi_n *} \quad (G\wedge *)$$

Proof. We prove here only $(G\wedge *)$ by induction on n. In the base case when $n = 1$ there is nothing to prove as $(G\wedge *)$ is the basic split rule $(\wedge)$. In the inductive case when $n + 1 \geq 2$ we consider an expansion by the rule:



We replace the rule $(G\wedge *)$ by $(\wedge *)$ followed on the right by an n-ary $(G\wedge *)$ rule which is admissible by IH. The new tableau is as follows:

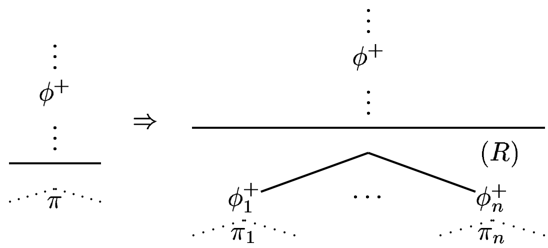


□

4.4 Inversion of expansion rules. Let

$$\frac{\phi^+}{\phi_1^+ \mid \cdots \mid \phi_n^+} \quad (R)$$

be an expansion rule with $n \geq 1$. The rule R can be an inversion or split rule as well as an identity or quantifier rule which will be introduced later. We say that the rule (R) is *invertible* if for every sequence of signed formulas Δ such that $\pi : \vdash_p [\Delta, \phi^+]$ there are tableaux π_i for $1 \leq i \leq n$ such that $\pi_i : \vdash_p [\Delta, \phi_i^+]$. Moreover the rule (R) is not applied with ϕ^+ as a premise, and the formula ϕ^+ is not used for the closing of a branch, in neither of tableaux π_i. The inversion of the rule R can be visualized as follows:

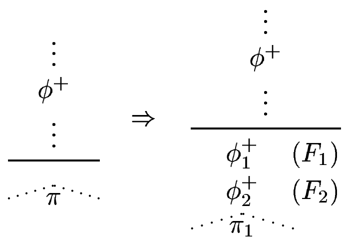


where we can in effect assume that the first expansion in the closed tableau on the right is by the rule (R) and that the premise ϕ^+ is used only once as indicated. This also means that the premise ϕ^+ is not used to close a branch.

Propositional flatten rules have the following schematic form:

$$\frac{\phi^+}{\phi_1^+}\,(F_1) \qquad \frac{\phi^+}{\phi_2^+}\,(F_2)\ .$$

We say that the flatten rules (F_1) and (F_2) are *invertible* if from $\pi\colon \vdash_p [\Delta, \phi^+]$ we can form a tableau π_1 such that $\pi_1\colon \vdash_p [\Delta, \phi_1^+, \phi_2^+]$ where the rules (F_1) and (F_2) are not used in π_1 with ϕ^+ as a premise and neither is the formula ϕ^+ used for closing of a branch. The inversion can be visualized as follows:

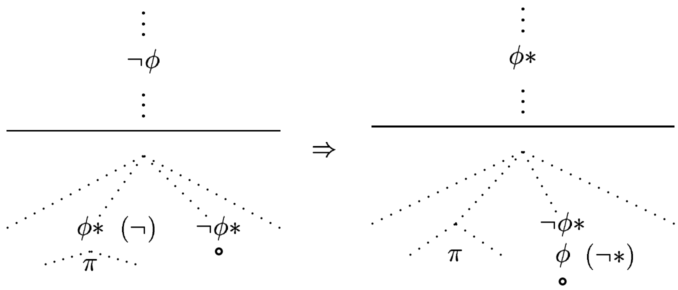


where we can in effect assume that the first two expansions in the closed tableau on the right are by the rules (F_1) and (F_2) and that the premise ϕ^+

is used only once as indicated. This also means that the premise ϕ^+ is not used to close a branch.

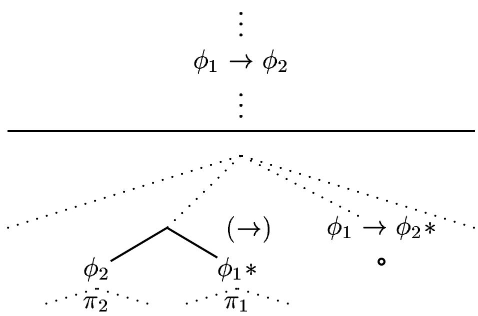
4.5 Inversion theorem. *Inversion, split, and flatten rules are invertible in propositional tableaux.*

Proof. The following diagram illustrates the inversion of a $(\neg)$ inversion rule:

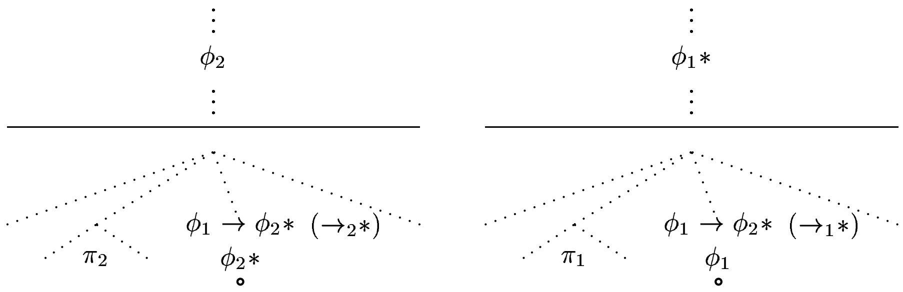


The closed tableau on the left shows just two of possibly many uses of the premise $\neg\phi$. The first one uses the premise in a $(\neg)$ rule and the second one closes the branch with $\neg\phi*$. The closed tableau on the right shows the inversion where the first used formula has been removed (because it is now in the top sequence) and the second one is now used as a premise of a $(\neg*)$ rule after which the branch closes. The two indicated transformations should be applied to all uses of the premise $\neg\phi$. The inversion of $\neg*$ inversion rules is similar.

The following diagram shows a closed tableau before the inversion of a $(\rightarrow)$ split rule:

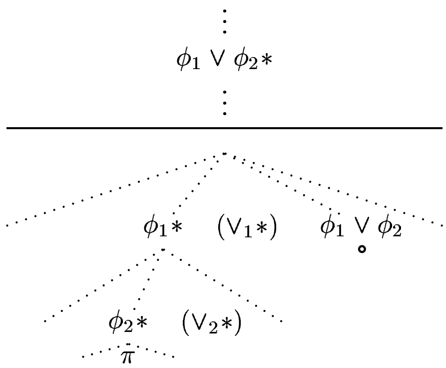


We have indicated two possible uses of the premise $\phi_1 \rightarrow \phi_2$. The first use is in a $(\rightarrow)$ rule and the second use is in the closure of a branch. We form the two inverted tableux as follows:

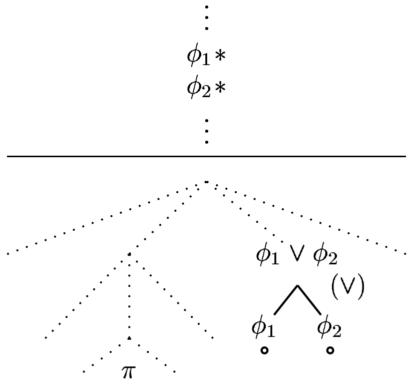


The expansions by the $(\to)$ rule have been replaced in the inverted tableau by tableaux π_2 and π_1 respectively and the closing formulas $\phi_1 \to \phi_2*$ have been expanded by the corresponding flatten rules after which the branches close. The inversion of other split rules is similar.

The following diagram shows a tableau before the inversion of a $(\vee_i*)$ flatten rules:



We have indicated three possible uses of the premise $\phi_1 \vee \phi_2*$. The first use is in a $(\vee_1*)$ rule after which there is a use in a $(\vee_2*)$ rule and third use is in the closure of a branch. We form the closed inverted tableau as follows:



In the inverted tableau the expansions by the $(\vee_i)$ rules have been removed and the closing formulas $\phi_1 \vee \phi_2$ have been expanded by $(\vee)$ split rules after which the branches close. The inversion of other flatten rules is similar. □

4.6 Cut rules. For any formula π the following is a *cut rule on* ϕ:

$$\frac{}{\phi \mid \phi*} \quad (C) \ .$$

That cut rules are admissible in propositional tableaux, i.e. that

$$\pi_1 \colon \vdash_p [\Delta, \phi] \text{ and } \pi_2 \colon \vdash_p [\Delta, \phi*] \Rightarrow \ \vdash_p [\Delta] \ .$$

holds can be proved by the following semantic argument. Assume $\pi_1 \colon \vdash_p [\Delta, \phi]$ and $\pi_2 \colon \vdash_p [\Delta, \phi*]$ and let $\bigwedge\Lambda \to \bigvee\Gamma$ be a sequent associated with Δ. By the Soundness lemma (3.12) we have $\models_p \phi \wedge \bigwedge\Lambda \to \bigvee\Gamma$ and $\models_p \bigwedge\Lambda \to \phi \vee \bigvee\Gamma$. By the truth table method we can see that also

$$\models_p (\phi \wedge \bigwedge\Lambda \to \bigvee\Gamma) \wedge (\bigwedge\Lambda \to \phi \vee \bigvee\Gamma) \to \bigwedge\Lambda \to \bigvee\Gamma$$

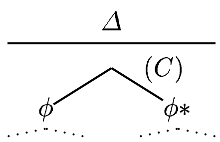
holds and hence $\models_p \bigwedge\Lambda \to \bigvee\Gamma$. By the Completeness lemma (3.13) we then get $\vdash_p [\Delta]$.

Unfortunately, this argument which depends on the soundness and completeness of propositional tableaux, cannot be extended to tableaux with quantifier rules without first proving the soundness and completeness theorem for such tableaux. Our intention is to reduce the quantificational logic to the propositional logic and to obtain the soundness and completeness of quantificational tableaux through the reduction and so the above semantical argument proving the admissibility of cut rules cannot be used.

We will now prove the admissibility of cuts by a syntactic (proof-theoretic) argument in a way which extends to the quantificational case.

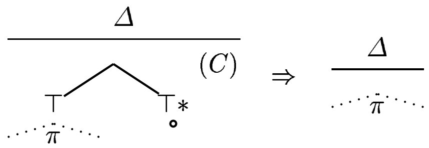
4.7 Lemma (Admissibility of cuts on propositional formulas). *If the cut rules on all propositional atoms in a formula ϕ are admissible in propositional tableaux then also the cut rule on ϕ is admissible.*

Proof. Assume that the cuts on the propositional atoms in $FPA(\phi)$ are admissible and prove by induction on the structure of ϕ that the cut on ϕ is admissible. We perform the case analysis of ϕ used in a cut as follows:



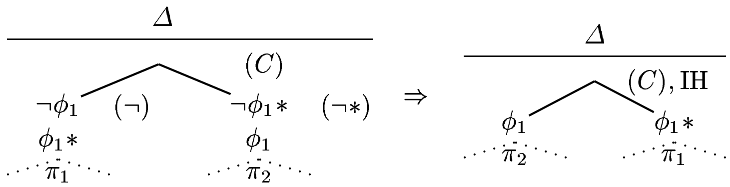
If ϕ is a propositional atom then the cut on ϕ is admissible from the assumption.

If $\phi \equiv \top$ then the expansion by the cut on $\top$ is shown in the following on the left:



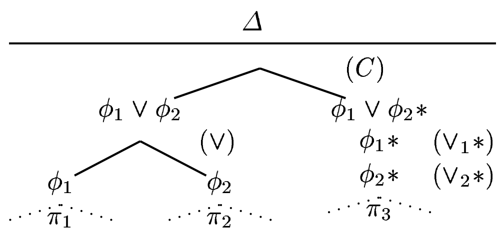
The assumption $\top$ is not used for anything in π and so the tableau π for Δ is closed as shown on the right. The case $\phi \equiv \bot$ is similar.

If $\phi \equiv \neg\phi_1$ then the expansion by the cut on $\neg\phi_1$ is shown in the following on the left:

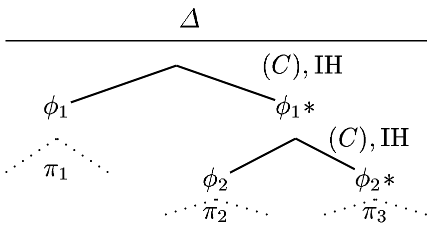


where we may assume without loss of generality that both inversion rules have been inverted. As a consequence the assumption $\neg\phi_1$ is not used for anything in π_1 and the goal $\neg\phi_1 *$ is not used in π_2. We transform the tableau as shown on the right where the cut on $\neg\phi_1$ has been removed and the inversion rules replaced by a cut on ϕ_1 which is admissible by IH.

If $\phi \equiv \phi_1 \vee \phi_2$ then the expansion by the cut on $\phi_1 \vee \phi_2$ is as follows:



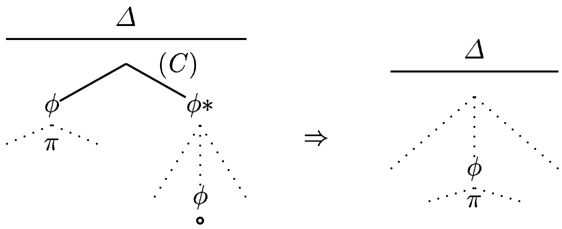
where we may assume without loss of generality that the $(\vee)$ split rule on the left and the $(\vee_i*)$ flatten rules on the right have been inverted. As a consequence the assumption $\phi_1 \vee \phi_2$ is not used for anything in π_1 and π_2 and the goal $\phi_1 \vee \phi_2*$ is not used in π_3. We transform the tableau as follows:



where the formula ϕ has been removed and the disjunctive rules replaced by two cuts on ϕ_1 and ϕ_2 respectively which are admissible by IH. The cases when the main propositional connective of ϕ is $\wedge$, $\rightarrow$, or $\leftrightarrow$ are similar. □

4.8 Lemma (Admissibility of cut rules on propositional atoms). *Cut rules on propositional atoms are admissible in propositional tableaux.*

Proof. Consider a closed propositional tableaux with a cut on a propositional atom ϕ shown in the following on the left:



We have shown in the tableau for $\Delta, \phi*$ one of possibly many branches closed on the pair of propositional atoms $\phi, \phi*$. Because the goal $\phi*$ cannot be used in propositional tableaux for anything else we can form a closed tableau for Δ shown on the right where we perform the indicated transformation for all branches closed on $\phi, \phi*$. □

4.9 Theorem (Admissibility of cuts). *Cut rules are admissible in propositional tableaux on arbitrary formulas.*

Proof. This is a direct consequence of Lemmas 4.7 and 4.8. □

4.10 Theorem (Lemma rule). *If $\vdash_p \phi$ then the following unary* lemma *rule*

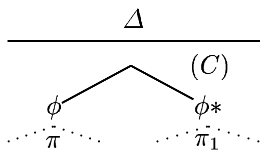
$$\frac{}{\phi} \quad (L)$$

is admissible in propositional tableaux, i.e. for any closed propositional tableau for Δ such that

$$\begin{array}{c} \Delta \\ \hline \phi \qquad (L) \\ \vdots \\ \pi \end{array} \tag{1}$$

we have $\vdash_p [\Delta]$.

Proof. Assume $\pi_1 \colon \vdash_p \phi$ and that the tableau (1) is closed and form the following closed tableau witnessing $\vdash_p [\Delta]$:



□

5 Tautological Consequence

Important extension of the notion of tautology is the notion of *tautological consequence* where we ask whether a formula ϕ follows from a finite or infinite set of formulas T by the laws of propositional logic alone.

Semantics

5.1 Tautological consequence. Fix a language $\mathcal{L}$ and let T be a set of formulas from $\mathcal{L}$. We define the relation *formula ϕ is a tautological consequence of T*, in symbols $T \models_p \phi$ as follows:

$$T \models_p \phi \Leftrightarrow (\mathcal{M} \models T \Rightarrow \mathcal{M} \models \phi) \text{ for all propositional interpretations } \mathcal{M}.$$

From this we can see that for $T = \emptyset$ we have $\emptyset \models_p \phi \Leftrightarrow \models_p \phi$.

Tautological consequence is a generalization of implication where we so to speak permit infinite many formulas in antecedent. If T is finite then we have $T \models_p \phi \Leftrightarrow \models_p \bigwedge T \to \phi$ by Lemma 5.2. If the set T is infinite then it can contain infinitely many propositional atoms and we cannot replace infinite propositional interpretations by finite ones as we did in the truth table method. Thus it seems that that the testing of $T \models_p \phi$ is a hard problem having to do with quantification over uncountably many propositional interpretations. Fortunately, this kind of quantification can be replaced by existential quantification over countably many finite sets of formulas. This is a consequence of the fundamental theorem 5.4.

5.2 Semantic deduction lemma. *For every finite set S of formulas we have:*

$$S \models_p \phi \Leftrightarrow\ \models_p \bigwedge S \to \phi \ .$$

Proof. We have $\mathcal{M} \models S$ iff $\mathcal{M} \models \psi$ for all $\psi \in S$ iff $\mathcal{M} \models \bigwedge S$. Thus $S \models_p \phi$ iff $\mathcal{M} \models S \Rightarrow \mathcal{M} \models_p \phi$ for all $\mathcal{M}$ iff $\mathcal{M} \models \bigwedge S \Rightarrow \mathcal{M} \models_p \phi$ for all $\mathcal{M}$ iff $\models_p \bigwedge S \to \phi$. □

5.3 Semantic weakening lemma. *If $S \subseteq T$ then*

$$S \models_p \phi \Rightarrow T \models_p \phi \ .$$

Proof. Assume $S \models_p \phi$ and take any propositional interpretation such that $\mathcal{M} \models T$ holds. Since this means $\mathcal{M} \models \phi$ for all $\phi \in T$ we also have $\mathcal{M} \models S$ and thus $\mathcal{M} \models \phi$ from the assumption. □

5.4 Compactness theorem.

$$T \models_p \phi \Rightarrow S \models_p \phi \text{ for a finite } S \subseteq T.$$

Proof. Assume $T \models_p \phi$. T is a set of formulas of a first order language $\mathcal{L}$ which has countably many formulas. Thus the set T is at most countable and we can write it in a form $T = \bigcup_{i \in N} S_i$ where each of the sets S_i is finite and we have $S_0 = \emptyset$ and $S_i \subseteq S_{i+1}$ (This is trivial if T is finite; otherwise, for instance, enumerate T and define S_i to be the set of the first i formulas in the enumeration).

We assign by the function $W(\mathcal{M})$ to every propositional interpretation $\mathcal{M}$ one of the finite sets S_i by:

$$W(\mathcal{M}) = S_i \quad \text{where } i \text{ is the least such that } \mathcal{M} \models S_i \Rightarrow \mathcal{M} \models \phi.$$

This is a legal definition because if $\mathcal{M} \models \phi$ then we have $\mathcal{M} \models S_0 \Rightarrow \mathcal{M} \models \phi$ and if $\mathcal{M} \not\models \phi$ then we have $\mathcal{M} \not\models T$. Thus $\mathcal{M} \not\models \psi$ for some $\psi \in T$ and, since then $\psi \in S_i$ for some i, we have $\mathcal{M} \not\models S_i$ for some i and there is a least such i. Define the set S to satisfy:

$$S := \bigcup \{W(\mathcal{M}) \mid \mathcal{M} \text{ is a propositional interpretation.}\}$$

We clearly have $S \subseteq T$. We also have $S \models_p \phi$ because for any propositional interpretation $\mathcal{M}$ such that $\mathcal{M} \models S$ we have $W(\mathcal{M}) \subseteq S$ and so $\mathcal{M} \models W(\mathcal{M})$. We then obtain $\mathcal{M} \models \phi$ from the definition of $W(\mathcal{M})$. Thus the theorem will be proved if we demonstrate by an indirect proof that the set S is finite.

So suppose that S is infinite. We will construct an increasing chain of propositional interpretations $\mathcal{M}_i$ in which the infiniteness of S will be propagated. Enumerate towards that end all propositional atoms of $\mathcal{L}$ in an infinite sequence $\psi_1, \psi_2, \psi_3, \ldots$ and define the finite sets of propositional atoms A_i for $i \in \mathbb{N}$ by

$$A_i = \bigcup_{1 \leq j \leq i} \{\psi_j\} \ .$$

Note that $A_0 = \emptyset$. Define an infinite sequence $\{\mathcal{M}_i\}_{i \in \mathbb{N}}$ of finite propositional interpretations $\mathcal{M}_i$ to satisfy:

$$\mathcal{M}_0 = \emptyset$$

$$\mathcal{M}_{i+1} = \begin{cases} \mathcal{M}_i & \text{if } \bigcup\{W(\mathcal{M}) \mid \mathcal{M} \cap A_{i+1} = \mathcal{M}_i\} \text{ is infinite} \\ \mathcal{M}_i \cup \{\psi_{i+1}\} & \text{otherwise.} \end{cases}$$

We clearly have $\mathcal{M}_i \subseteq \mathcal{M}_{i+1}$. Define the sets I_i for $i \in \mathbb{N}$ to satisfy:

$$I_i = \bigcup\{W(\mathcal{M}) \mid \mathcal{M} \cap A_i = \mathcal{M}_i\}$$

and prove by induction on i:

$$M_i \subseteq A_i \text{ and } I_i \text{ is infinite.}$$

In the base case we have $\mathcal{M}_0 = \emptyset = A_0$ and the set

$$I_0 = \bigcup\{W(\mathcal{M}) \mid \mathcal{M} \cap A_0 = \mathcal{M}_0\} = \bigcup\{W(\mathcal{M}) \mid \emptyset = \emptyset\} = S \ .$$

is infinite. In the inductive case we have

$$\mathcal{M}_{i+1} \subseteq \mathcal{M}_i \cup \{\psi_{i+1}\} \overset{\text{IH}}{\subseteq} A_i \cup \{\psi_{i+1}\} = A_{i+1} \ .$$

We note that if $\mathcal{M} \cap A_i = \mathcal{M}_i$ then

$$\begin{aligned} \mathcal{M} \cap A_{i+1} =& \mathcal{M} \cap (A_i \cup \{\psi_{i+1}\}) = (\mathcal{M} \cap A_i) \cup (\mathcal{M} \cap \{\psi_{i+1}\}) = \\ & \mathcal{M}_i \cup (\mathcal{M} \cap \{\psi_{i+1}\}) = \begin{cases} \mathcal{M}_i & \text{if } \psi_{i+1} \notin \mathcal{M} \\ \mathcal{M}_i \cup \{\psi_{i+1}\} & \text{if } \psi_{i+1} \in \mathcal{M} \end{cases} \end{aligned}$$

and so

$$\begin{aligned} I_n =& \bigcup\{W(\mathcal{M}) \mid \mathcal{M} \cap A_i = \mathcal{M}_i\} = \\ & \bigcup\{W(\mathcal{M}) \mid \mathcal{M} \cap A_{i+1} = \mathcal{M}_i\} \cup \bigcup\{W(\mathcal{M}) \mid \mathcal{M} \cap A_{i+1} = \mathcal{M}_i \cup \{\psi_{i+1}\}\} \ . \end{aligned}$$

One of the two sets on the right must be infinite because I_n is by IH. We consider two cases. If the first set is infinite then $\mathcal{M}_{i+1} = \mathcal{M}_i$ by definition and the first set is I_{n+1}. If the first set is finite then $\mathcal{M}_{i+1} = \mathcal{M}_i \cup \{\psi_{i+1}\}$

and so the second set, which must be infinite, is I_{n+1}. Thus in both cases I_{n+1} is an infinite set.

We construct a propositional interpretation $\mathcal{M} = \bigcup\{\mathcal{M}_i \mid i \in \mathbb{N}\}$ and we have $W(\mathcal{M}) = S_k$ for the least k such that $\mathcal{M} \models S_k \Rightarrow \mathcal{M} \models \phi$. Let i be the least number such that $FPA(S_k \cup \{\phi\}) \subseteq A_i$ and let $\mathcal{N}$ be any propositional interpretation such that $\mathcal{N} \cap A_i = \mathcal{M}_i$. We have

$$\mathcal{N}^{S_k \cup \{\phi\}} = \mathcal{N} \cap FPA(S_k \cup \{\phi\}) = \mathcal{N} \cap A_i \cap FPA(S_k \cup \{\phi\}) = \mathcal{M}_i \cap FPA(S_k \cup \{\phi\}) = \mathcal{M}_i^{S_k \cup \{\phi\}}$$

and so $\mathcal{N}$ and $\mathcal{M}_i$ are $S_k \cup \{\phi\}$-equivalent by the Equivalence lemma (see 3.6). From this we get $W(\mathcal{N}) = W(\mathcal{M}_i)$ and hence

$$I_i = \bigcup\{W(\mathcal{N}) \mid \mathcal{N} \cap A_i = \mathcal{M}_i\} = \bigcup\{W(\mathcal{M}_i) \mid \mathcal{N} \cap A_i = \mathcal{M}_i\} = W(\mathcal{M}_i)$$

which is a contradiction because the set $W(\mathcal{M}_i)$ is finite (the set is actually equal to S_k). □

5.5 Remark. The reader familiar with set theory will recognize the construction of the sequence $\{\mathcal{M}_i\}_{i\mathbb{N}}$ in the proof of the Compactness theorem as the construction of an infinite branch of an infinite tree with finitely branching nodes in the proof of the König's lemma. The lemma says that in every finitely branching tree with an infinite number of nodes there is an infinite branch.

5.6 Semidecidability of tautological consequence. Let $\mathcal{L}$ be a language and T an infinite decidable set of its formulas. We have noted in Par. 5.1 that in order to decide the unary relation *ϕ is a tautological consequence of T*, i.e. $T \models_p \phi$, one would expect to test $\mathcal{M} \models T \Rightarrow \mathcal{M} \models \phi$ for uncountably many propositional interpretations $\mathcal{M}$. By the Compactness theorem and Lemma 5.3 we however know that it is sufficient to decide whether $S \models_p \phi$ holds for a finite subset S of T. As there are countably many finite subsets S of the countable T we have to test by Lemma 5.2 countably many times whether $\bigwedge S \to \phi$ is a tautology.

One of the ways to organize the tests is as follows. We encode all formulas of $\mathcal{L}$ into $\mathbb{N}$. Starting from 0 we then successively test for all natural numbers i whether i is a list such that for every $j \in i$ (there are finitely many such j's) the number j codes a formula ϕ (this can be effectively done). If so, then we test whether $\phi \in T$. If this holds for all $j \in i$ we can effectively form the finite set S of all formulas of $\mathcal{L}$ coded by i. We then test whether $\bigwedge S \to \phi$ is a tautology. If this is the case we stop the testing and we know that ϕ is a tautological consequence of T. If not then we have to continue with the next number $i + 1$ and if ϕ is not a tautological consequence then by Lemmas 5.3 and 5.2 there is no finite $S \subset T$ such that $\bigwedge S \to \phi$ is a tautology and we will never discover the fact.

Predicates $P(x)$ such that if there is an x satisfying P we can effectively find it but we go on searching forever if for all x not $P(x)$ are called *semidecidable* predicates. It can be shown by the methods of recursion theory that the unary predicate $T \models_p \phi$ is semidecidable and that the above described method of crude search cannot be improved upon. Later in this text we will show how to reduce the general questions of logical validity of formulas and of logical consequence to the questions of tautological consequence from decidable sets. This will mean that the best we can do in first order logic is to find a proof of a formula from given axioms if there is one. If the formula is unprovable we might never discover it.

Syntax

5.7 Axiom rules. Fix a language $\mathcal{L}$. We now extend propositional tableaux so we can prove that ϕ is a tautological consequence of a decidable set T of formulas $\mathcal{L}$. For every $\psi \in T$ we add a unary *axiom rule*:

$$\frac{\quad}{\psi} \quad (Ax)$$

which means that we can extend a branch of a tableau at an arbitrary position with the assumption ψ. Note that the decidability of T is crucial for this because we can use the axiom rule only if $\psi \in T$. Without decidability we would not be able to recognize whether a given tree of signed formulas is a legal tableau or not.

5.8 Proofs with propositional tableaux with axioms. A propositional tableau π for a sequence of signed formulas Δ which possibly uses axiom rules from T is a *propositional tableau for Δ from axioms T*. If π is closed then we assert this by writing $\pi : T \vdash_p [\Delta]$. Note that if $T = \emptyset$ then we have $\pi : \emptyset \vdash_p [\Delta] \Leftrightarrow \pi\colon \vdash_p [\Delta]$. When $\Delta \equiv \phi*$ then we say that π *propositionally proves ϕ from axioms T* and write it as $\pi : T \vdash_p \phi$. We use abbreviations similar to those discussed in Par. 3.11 also for the proofs from axioms.

5.9 Theorem (Admissible expansion rules in propositional tableaux with axioms). *All rules proved admissible for propositional tableaux in Sect. 4 are also admissible in propositional tableaux with axioms.*

Proof. Inspection of the proofs of admissibility of expansion rules in Sect. 4 reveals that the proofs remain correct also for tableaux with axiom rules because their presence does not affect the proofs of admissibility. □

5.10 Syntactic weakening lemma. *If $S \subseteq T$ then*

$$S \vdash_p \phi \Rightarrow T \vdash_p \phi \;.$$

Proof. Assume $\pi : S \vdash_p \phi$. Every every axiom rule for $\psi \in S$ used in π is also an axiom rule for $\psi \in T$ and so $\pi : T \vdash_p \phi$. □

5.11 Syntactic compactness lemma.

$$T \vdash_p \phi \Rightarrow S \vdash_p \phi \text{ for a finite } S \subseteq T.$$

Proof. Assume $\pi : T \vdash_p \phi$ and construct the finite set S to consist of all conclusions ψ of axiom rules for $\psi \in T$ used in the tableau π. The same rules are axiom rules for $\psi \in S$ and so $\pi : S \vdash_p \phi$. □

5.12 Deduction theorem. For a finite set of formulas S:

$$S \vdash_p \phi \Leftrightarrow \; \vdash_p \bigwedge S \to \phi \; .$$

Proof. Let $S = \{\psi_1, \ldots, \psi_n\}$. In the direction $(\Rightarrow)$ assume $\pi : S \vdash_p \phi$, i.e. $\pi : S \vdash_p [\phi*]$. If $n = 0$ then $\bigwedge S \equiv \top$ and we derive $\vdash_p \top \to \phi$ as follows:

$$\begin{array}{ll} \top \to \phi* & \\ \hline \phi* & (\to_2 *) \\ \quad\cdots\pi\cdots & \end{array}$$

If $n > 0$ then denote by π_1 the tableau formed from π by deleting all expansions by axiom rules. We clearly have $\pi_1 \colon \vdash_p [\phi*, \psi_1, \ldots, \psi_n]$. which is shown in the following on the left:

$$\begin{array}{c} \phi* \\ \psi_1 \\ \vdots \\ \psi_n \\ \hline \cdots\pi_1\cdots \end{array} \quad \Rightarrow \quad \begin{array}{cl} \psi_1 \wedge \cdots \wedge \psi_n \to \phi* & \\ \hline \phi* & (\to_2 *) \\ \psi_1 \wedge \cdots \wedge \psi_n & (\to_1 *) \\ \psi_1 & (G\wedge_1) \\ \vdots & \\ \psi_n & (G\wedge_n) \\ \cdots\pi_1\cdots & \end{array}$$

We form the closed tableau on the right which is expanded with the help of the generalized flatten rules (see Thm. 4.2) and so $\vdash_p \bigwedge S \to \phi$.

In the direction $(\Leftarrow)$ assume $\pi \colon \vdash_p [\psi_1 \wedge \cdots \wedge \psi_n \to \phi*]$. We can assume by inversion that the first two expansions in π are by $(\to_i *)$ flatten rules such that the goal $\bigwedge S \to \phi*$ is not used in π_1. If $n = 0$ then the situation is shown on the left:

$$\begin{array}{cl} \top \to \phi* & \\ \hline \phi* & (\to_2 *) \\ \top & (\to_1 *) \\ \cdots\pi_1\cdots & \end{array} \quad \Rightarrow \quad \begin{array}{c} \phi* \\ \hline \cdots\pi_1\cdots \end{array}$$

and we can form a closed tableau π for $\phi*$ because the assumption $\top$ cannot be used in π_1. If $n > 0$ then we can assume that the inversion of $(\to_i *)$ flatten rules is followed by by $n-1$ inversions of $(\wedge)$ flatten rules which is shown in the following on the left:

$$
\begin{array}{c}
\psi_1 \wedge \cdots \wedge \psi_n \to \phi* \\
\hline
\phi* \quad (\to_2 *) \\
\psi_1 \wedge \cdots \wedge \psi_n \quad (\to_1 *) \\
\psi_1 \quad (\wedge_1) \\
\psi_2 \wedge \cdots \wedge \psi_n \quad (\wedge_2 *) \\
\vdots \\
\psi_n \quad (\wedge_2) \\
\cdots \pi_1 \cdots
\end{array}
\quad \Rightarrow \quad
\begin{array}{c}
\phi* \\
\hline
\psi_1 \quad (Ax) \\
\cdots \\
\psi_n \quad (Ax) \\
\cdots \pi_1 \cdots
\end{array}
$$

We form the closed tableau on the right which is expanded with the help of axiom rules from S and in which we have omitted the assumptions $\psi_i \wedge \cdots \wedge \psi_n$ for $1 \leq i < n$ because they cannot be used in π_1. We thus have $S \vdash_p \phi$. □

5.13 Corollary (Soundness and completeness of propositional tableaux with axioms). *For any fromula ϕ and set of formulas T there is in each direction a finite subset S of T such that the following holds:*

$$
\begin{array}{ccc}
T \models_p \phi & & T \vdash_p \phi \\
\Uparrow 5.3 \Downarrow 5.4 & & \Uparrow 5.10 \Downarrow 5.11 \\
S \models_p \phi & & S \vdash_p \phi \\
\Uparrow\!\Downarrow 5.2 & & \Uparrow\!\Downarrow 5.12 \\
\models_p \bigwedge S \to \phi & \Leftarrow 3.12 \Rightarrow 3.13 & \vdash_p \bigwedge S \to \phi
\end{array}
$$

□

5.14 Semidecidability of tautological consequence revisited. One of the consequences of the Corollary 5.13 is that we can semidecide the relation $T \models_p \phi$ by tableaux instead of testing for tautologies as outlined in Par. 5.6. We do this by constructing possibly infinite branches which are *axiomatically complete*, i.e. which apply the axiom rules for all axioms from T.

In order to decide the relation $T \models_p \phi$ we enumerate the axioms of T into a sequence $\psi_1, \psi_2, \ldots,$ and construct a propositional tableau for the goal $\phi*$. We select an open propositionally complete branch of it (if any) and we extend it by an axiom rule by taking ψ_1 into assumptions. We then construct a propositional tableau under the assumption. We select a not closed propositionally complete branch again (if any) and expand it with the axiom rule for ψ_2. We continue in this way in the hope of closing the branch. If this happens we apply the same procedure of systematically applying one axiom after another to the remaining open branches. If all branches close then $T \models_p \phi$ holds.

Otherwise, if T is finite we stop with at least one open branch Δ which is both propositionally and axiomatically complete. The branch is finite. If T is infinite we will go on extending a branch forever because there is at least one infinite branch which is open and propositionally and axiomatically complete.

In both cases there is a propositional interpretation $\mathcal{M}$ constructed by collecting all propositional atoms in the assumptions. We have proved in Par. 3.15 that we have $\mathcal{M} \not\models \phi$ and $\mathcal{M} \models \phi_1$ for every assumption ϕ_1 in the branch. Since the branch contains all axioms T as assumptions we have in particular $\mathcal{M} \models T$. But this means that we have $T \not\models_p \phi$. Note that we can effectively determine this fact only when T is finite.

6 Quasitautological Consequence

In this chapter we investigate formulas always true on the strength of propositional logic and of properties of identity.

Language

6.1 Free terms. *Free terms* of a set of formulas S of a language $\mathcal{L}$ are terms occurring outside of quantifiers in the formulas of S. This is made precise by a metamathematical function $FT(\alpha)$ defined on terms, formulas, and sets of formulas to yield the set of free terms of α. The function FT satisfies:

$$
\begin{aligned}
FT(x) &= \{x\} \\
FT(f(\tau_1, \ldots, \tau_n)) &= \{f(\tau_1, \ldots, \tau_n)\} \cup FT(\tau_1) \cup \ldots \cup FT(\tau_n) \\
FT(\tau_1 = \tau_2) &= FT(\tau_1) \cup FT(\tau_2) \\
FT(P(\tau_1, \ldots, \tau_n)) &= FT(\tau_1) \cup \ldots \cup FT(\tau_n) \\
FT(\forall x \phi) &= \emptyset \\
FT(\exists x \phi) &= \emptyset \\
FT(\top) &= \emptyset \\
FT(\bot) &= \emptyset \\
FT(\neg \phi) &= FT(\phi) \\
FT(\phi_1 \vee \phi_2) &= FT(\phi_1) \cup FT(\phi_2) \\
FT(\phi_1 \wedge \phi_2) &= FT(\phi_1) \cup FT(\phi_2) \\
FT(\phi_1 \rightarrow \phi_2) &= FT(\phi_1) \cup FT(\phi_2) \\
FT(\phi_1 \leftrightarrow \phi_2) &= FT(\phi_1) \cup FT(\phi_2) \\
FT(T) &= \bigcup \{FT(\phi) \mid \phi \in T\} \ .
\end{aligned}
$$

Semantics

6.2 Structures. A structure $\mathcal{A}$ for a language $\mathcal{L}$ is given by

1. a non-empty set D, called the *domain* of the structure,
2. for every n-ary function symbol f of $\mathcal{L}$ an n-ary function $f^{\mathcal{A}}$ over D, i.e. $f^{\mathcal{A}} : D^n \mapsto D$, called the *interpretation of* f,
3. for every n-ary predicate symbol P of $\mathcal{L}$ a subset $P^{\mathcal{A}}$ of D^n called the *interpretation of* P.

Here we define $D^0 = \{\emptyset\}$ and identify nullary functions over D with elements of D. Thus for every constant c of $\mathcal{L}$ we have $c^{\mathcal{A}} \in D$ and for every propositional constant P of $\mathcal{L}$ we have $P^{\mathcal{A}} \subseteq \{\emptyset\}$. We agree to identify the value $P^{\mathcal{A}} = \emptyset$ with falsehood and the value $P^{\mathcal{A}} = \{\emptyset\}$ with truth.

Structures are mathematical objects (triples of sets) whose purpose is to assign meaning to terms and formulas of $\mathcal{L}$. A structure $\mathcal{A}$ is *finite* if its domain D is a finite set and *infinite* otherwise. $\mathcal{A}$ is a *numeric* structure if its domain is a subset of natural numbers.

6.3 Assignments. For a structure $\mathcal{A}$ for $\mathcal{L}$ with the domain D we call a function a from $\mathbb{N}$ to D *assignment in* $\mathcal{A}$. The idea is that the assignment a assigns the value $a(i) \in D$ to the variable v_i.

6.4 Identity interpretations. An *identity interpretation* $\mathcal{M}$ *for* $\mathcal{L}$ is a triple $\langle Q, \mathcal{A}, a\rangle$ where Q, called the *quantifier set*, is a subset of quantifier formulas of $\mathcal{L}$, $\mathcal{A}$ is a structure for $\mathcal{L}$ and a is an assignment in $\mathcal{A}$. An identity $\mathcal{M}$ is *finite* if its structure is finite and *numeric* if its structure is numeric.

Identity interpretations uniquely determine the meaning of terms and formulas as shown in the following two paragraphs.

6.5 Denotation of terms. For a given identity interpretation $\mathcal{M} = \langle Q, \mathcal{A}, a\rangle$ for $\mathcal{L}$ with D the domain of $\mathcal{A}$ we assign to every term τ of $\mathcal{L}$ its *denotation*, designated as $\tau^{\mathcal{M}}$, to be the element of D satisfying the following:

$$\begin{aligned} v_i^{\mathcal{M}} &= a(i) \\ f(\tau_1, \ldots, \tau_n)^{\mathcal{M}} &= f^{\mathcal{A}}(\tau_1^{\mathcal{M}}, \ldots, \tau_n^{\mathcal{M}}) \quad f \text{ is } n\text{-ary function symbol of } \mathcal{L}. \end{aligned}$$

If f is a constant symbol, i.e. a nullary function symbol, then we abbreviate $f^{\mathcal{A}}()$ to $f^{\mathcal{A}}$.

The identity interpretation $\mathcal{M}$ is *canonical* if it is a numeric interpretation and for every element d in the domain of its structure we have $d = \tau^{\mathcal{M}}$ for some term τ.

6.6 Satisfaction relation for identity interpretations. Let $\mathcal{M} = \langle Q, \mathcal{A}, a\rangle$ be an identity interpretation for $\mathcal{L}$. For every formula ϕ of $\mathcal{L}$ we define the unary relation $\mathcal{M} \models \phi$, read as $\mathcal{M}$ *satisfies* ϕ, to be similar to the satisfaction relation for propositional interpretations (see Par. 3.3) when ϕ is a propositional formula. If ϕ is a propositional atom then we define

$$\begin{aligned} \mathcal{M} \models \forall x\phi &\Leftrightarrow \forall x\phi \in Q \\ \mathcal{M} \models \exists x\phi &\Leftrightarrow \exists x\phi \in Q \\ \mathcal{M} \models \tau_1 = \tau_2 &\Leftrightarrow \tau_1^{\mathcal{M}} = \tau_2^{\mathcal{M}} \\ \mathcal{M} \models P(\tau_1, \ldots, \tau_n) &\Leftrightarrow \langle \tau_1^{\mathcal{M}}, \ldots, \tau_n^{\mathcal{M}}\rangle \in P^{\mathcal{A}} \ . \end{aligned}$$

Here the 'zero-tuple' $\langle\rangle$ is defined as the empty set $\emptyset$. This means that for a propositional constant P we have $\mathcal{M} \models P$ iff $P^{\mathcal{A}} = \{\emptyset\}$. This should explain why we have identified the set $\{\emptyset\}$ with the truth.

Assignments in $\mathcal{M}$ are used only in atomic formulas which obtain the meaning from the interpretation of function and predicate symbols specified by the points (2) and (3) in Par. 6.2. Specifically, the meaning of identity $\tau_1 = \tau_2$ is determined as the identity of the denotations $\tau_1^{\mathcal{M}} = \tau_2^{\mathcal{M}}$. The reader will note that the symbol of identity in the formula $\tau_1 = \tau_2$ is just a symbol whereas the same symbol in $\tau_1^{\mathcal{M}} = \tau_2^{\mathcal{M}}$ stands for the relation of identity over the domain of $\mathcal{M}$. The meaning of quantifier formulas is determined from the quantifier set Q similarly as the meaning of propositional atoms is determined from propositional interpretations.

6.7 Lemma (Reduction to propositional interpretations). *To every identity interpretation $\mathcal{M}$ for $\mathcal{L}$ there is an equivalent propositional interpretation $\mathcal{N}$.*

Proof. Take an identity interpretation $\mathcal{M} = \langle Q, \mathcal{A}, a\rangle$ for $\mathcal{L}$. We construct the propositional interpretation $\mathcal{N}$ as follows:

$$\mathcal{N} = \{\psi \mid \mathcal{M} \models \psi \text{ for propositional atoms } \psi\} \ .$$

We prove the equivalence by induction on the construction of ϕ. If ϕ is a propositional atom then $\mathcal{N} \models \phi$ iff $\phi \in \mathcal{N}$ iff $\mathcal{M} \models \phi$. If $\phi \equiv \phi_1 \vee \phi_2$ then we have $\mathcal{N} \models \phi_1 \vee \phi_2$ iff $\mathcal{N} \models \phi_1$ or $\mathcal{N} \models \phi_2$ iff, by IH, $\mathcal{M} \models \phi_1$ or $\mathcal{M} \models \phi_2$ iff $\mathcal{M} \models \phi_1 \vee \phi_2$. The remaining cases are similar. □

6.8 Equality axioms. Let $\mathcal{L}$ be a first-order language. For all terms $\tau_1, \ldots,$ τ_n, $\rho_1, \ldots, \rho_n$ the following formulas, designated by Eq, are called *equality axioms for* $\mathcal{L}$:

$$\tau_1 = \tau_1 \tag{1}$$

$$\tau_1 = \tau_2 \to \tau_2 = \tau_1 \tag{2}$$

$$\tau_1 = \tau_2 \wedge \tau_2 = \tau_3 \to \tau_1 = \tau_3 \tag{3}$$

$$\begin{array}{c}\tau_1 = \rho_1 \wedge \cdots \wedge \tau_n = \rho_n \to f(\tau_1, \ldots, \tau_n) = f(\rho_1, \ldots, \rho_n) \\ f \text{ is } n\text{-ary function symbol, } n > 0\end{array} \tag{4}$$

$$\begin{array}{c}\tau_1 = \rho_1 \wedge \cdots \wedge \tau_n = \rho_n \wedge P(\tau_1, \ldots, \tau_n) \to P(\rho_1, \ldots, \rho_n) \\ P \text{ is } n\text{-ary predicate symbol, } n > 0.\end{array} \tag{5}$$

Formulas (1) are axioms of *reflexivity*, (2) are axioms of *symmetry*, (3) are axioms of *transitivity*, (4) and (5) are axioms of *function* and *predicate substitution* respectively.

We designate by Eq^T the *restriction* of equality axioms to the free terms of a set of formulas T, i.e.:

$$Eq^T = \{\phi \mid \phi \in Eq \wedge FT(\phi) \subseteq FT(T)\} \ .$$

6.9 Lemma (Expansion of propositional interpretations). *To every set of formulas T of $\mathcal{L}$ and every propositional interpretation $\mathcal{M}$ for $\mathcal{L}$ such that $\mathcal{M} \models Eq^T$ there is a T-equivalent canonical identity interpretation $\mathcal{N}$. $\mathcal{N}$ is finite if the set $FT(T)$ is finite.*

Proof. Take any T and any propositional interpretation $\mathcal{M}$ such that $\mathcal{M} \models Eq^T$. We wish to define the identity interpretation $\mathcal{N} = \langle Q, \mathcal{A}, a \rangle$. We define the quantifier set Q as follows:

$$Q = \{\psi \mid \psi \text{ is a quantifier formula and } \psi \in \mathcal{M} \ .\}$$

We enumerate the set $\mathcal{T} = FT(T)$ by a possibly finite or even empty sequence:

$$\sigma_0, \sigma_1, \sigma_2, \ldots$$

Clearly, the sequence has as many elements as is the cardinality of $\mathcal{T}$. We define the domain D of the structure $\mathcal{A}$ with the help of a *representant* function r mapping the terms of $\mathcal{L}$ to $\mathbb{N}$ by

$$r(\tau) = \min\{i \mid \tau \in \mathcal{T} \Rightarrow \mathcal{M} \models \tau = \sigma_i\} \ .$$

This is a legal definition only if the argument set to min is non-empty for every τ. That this is so can be seen by considering two cases. If $\tau \notin \mathcal{T}$ then the argument to min is $\mathbb{N}$ and so $r(\tau) = 0$. If $\tau \in \mathcal{T}$ then we have $\tau \equiv \sigma_i$ for some i and, since the reflexivity axiom $\tau = \sigma_i$ is in Eq^T, we have $\mathcal{M} \models \tau = \sigma_i$. Hence the argument to min contains i. We now define the domain D of $\mathcal{N}$ as the range of the function r:

$$D = \{r(\tau) \mid \tau \text{ is a term of } \mathcal{L}\} \ .$$

We have $D \subseteq \mathbb{N}$ and if $\mathcal{T} = \emptyset$ then $D = \{0\}$. Otherwise $\sigma_0 \in \mathcal{T}$ and $r(\sigma_0) = 0 \in D$. Note that D has no more elements than $\mathcal{T}$ if $\mathcal{T} \neq \emptyset$ is finite. We prove

$$\tau_1, \tau_2 \in \mathcal{T} \Rightarrow (r(\tau_1) = r(\tau_2) \Leftrightarrow \mathcal{M} \models \tau_1 = \tau_2) \ . \tag{1}$$

Assume $\tau_1, \tau_2 \in \mathcal{T}$. In the direction $(\Rightarrow)$ also assume $r(\tau_1) = r(\tau_2)$. From this we get $\mathcal{M} \models \tau_1 = \sigma_i$ and $\mathcal{M} \models \tau_2 = \sigma_i$ for some i. Since $\sigma_i \in \mathcal{T}$, the symmetry axiom $\tau_2 = \sigma_i \to \sigma_i = \tau_2$ and the transitivity axiom $\tau_1 = \sigma_i \to \sigma_i = \tau_2 \to \tau_1 = \tau_2$ are in the set Eq^T. Thus also $\mathcal{M} \models \tau_1 = \tau_2$ holds. In the direction $(\Leftarrow)$ assume $\mathcal{M} \models \tau_1 = \tau_2$. We have $r(\tau_1) = i$ for the least i such that $\sigma_i \in \mathcal{T}$ and $\mathcal{M} \models \tau_1 = \sigma_i$ holds. By appropriate symmetry and transitivity axioms in Eq^T we obtain $\mathcal{M} \models \tau_2 = \sigma_i$. We have $r(\tau_2) = j$ for the least $j \leq i$ such that $\sigma_j \in \mathcal{T}$ and $\mathcal{M} \models \tau_2 = \sigma_j$ holds. If $j < i$ then by symmetry and transitivity axioms in Eq^T we would get $\mathcal{M} \models \tau_1 = \sigma_j$ contradicting the definition of i. Hence $i = j$ and so $r(\tau_1) = r(\tau_2)$.

We now take any n-ary function symbol f of $\mathcal{L}$ and define the interpretation $f^{\mathcal{A}} : D^n \mapsto D$ of f as follows:

$$f^{\mathcal{A}}(d_1, \ldots, d_n) = r(f(\tau_1, \ldots, \tau_n)) \quad \text{where } r(\tau_1) = d_1, \ldots, r(\tau_n) = d_n.$$

We must convince ourselves first that this is a legal definition. If $n = 0$ then we have $f^{\mathcal{A}} = r(f)$. If $n > 0$ and $\mathcal{T} = \emptyset$ then, since $D = \{0\}$, we always have

$r(f(\tau_1,\ldots,\tau_n)) = 0$. If $n > 0$ and $\mathcal{T} \neq \emptyset$ then, since for every $d \in D$ we have $d = r(\rho)$ for some $\rho \in \mathcal{T}$, the definition determines $f^{\mathcal{A}}(d_1,\ldots,d_n)$ uniquely if the following holds:

$$r(\tau_1) = r(\rho_1) \wedge \cdots \wedge r(\tau_n) = r(\rho_n) \to r(f(\tau_1,\ldots,\tau_n)) = r(f(\rho_1,\ldots,\rho_n))$$

for every $\tau_1, \ldots, \tau_n, \rho_1, \ldots, \rho_n$ in $\mathcal{T}$. From the assumptions we obtain $\mathcal{M} \models \tau_1 = \rho_1, \ldots, \mathcal{M} \models \tau_n = \rho_n$ by (1). Since the function substitution axiom 6.8(4) is in Eq^T and $\mathcal{M} \models Eq^T$, we have $\mathcal{M} \models f(\tau_1,\ldots,\tau_n) = f(\rho_1,\ldots,\rho_n)$ and so $r(f(\tau_1,\ldots,\tau_n)) = r(f(\rho_1,\ldots,\rho_n))$ by (1).

For every n-ary predicate symbol P of $\mathcal{L}$ we define its interpretation $P^{\mathcal{A}} \subseteq D^n$:

$$P^{\mathcal{A}} = \{\langle r(\tau_1),\ldots,r(\tau_n)\rangle \mid \mathcal{M} \models P(\tau_1,\ldots,\tau_n) \text{ and } \tau_1,\ldots,\tau_n \in \mathcal{T}\} .$$

We prove

$$\rho_1,\ldots,\rho_n \in \mathcal{T} \Rightarrow (\mathcal{M} \models P(\rho_1,\ldots,\rho_n) \Leftrightarrow \langle r(\rho_1),\ldots,r(\rho_n)\rangle \in P^{\mathcal{A}}) \qquad (2)$$

by taking any $\rho_1,\ldots,\rho_n \in \mathcal{T}$. In the direction ($\Rightarrow$) the property follows from the definition of $P^{\mathcal{A}}$ (even when $n = 0$ because $\rho_1,\ldots,\rho_n \in \mathcal{T}$ holds vacuously and we then have $P^{\mathcal{A}} = \{\langle\rangle\} = \{\emptyset\}$). In the direction ($\Leftarrow$) assume $\langle r(\rho_1),\ldots,r(\rho_n)\rangle \in P^{\mathcal{A}}$. If $n = 0$ then this means $\langle\rangle \in P^{\mathcal{A}}$ and so $\mathcal{M} \models P$. If $n > 0$ we have $r(\tau_1) = r(\rho_1), \ldots, r(\tau_n) = r(\rho_n)$ and $\mathcal{M} \models P(\tau_1,\ldots,\tau_n)$ for some $\tau_1,\ldots,\tau_n \in \mathcal{T}$. We get $\mathcal{M} \models \tau_1 = \rho_1, \ldots, \mathcal{M} \models \tau_n = \rho_n$ from (1). Since the predicate substitution axiom 6.8(5) $\in Eq^T$ and $\mathcal{M} \models Eq^T$, we obtain $\mathcal{M} \models P(\rho_1,\ldots,\rho_n)$.

The structure $\mathcal{A}$ is now defined and we define the assignment $a : \mathbb{N} \mapsto D$ by $a(i) = r(v_i)$. Thus also the identity interpretation $\mathcal{N}$ is defined and we prove:

$$\tau \in \mathcal{T} \Rightarrow \tau^{\mathcal{N}} = r(\tau) \qquad (3)$$

by induction on the construction of the term τ. So assume $\tau \in \mathcal{T}$ and continue by the case analysis of τ. If $\tau \equiv v_i$ then $v_i^{\mathcal{N}} = a(i) = r(v_i)$. If $\tau \equiv f(\tau_1,\ldots,\tau_n)$ then

$$f(\tau_1,\ldots,\tau_n)^{\mathcal{N}} = f^{\mathcal{A}}(\tau_1^{\mathcal{N}},\ldots,\tau_n^{\mathcal{N}}) \stackrel{\text{IH}}{=} f^{\mathcal{A}}(r(\tau_1),\ldots,r(\tau_n)) = r(f(\tau_1,\ldots,\tau_n)) .$$

Note that $\mathcal{N}$ is a canonical interpretation because if $d \in D$ then $d = r(\tau) = \tau^{\mathcal{N}}$ for some τ.

The lemma will be proved when we prove for every formula $\phi \in T$ the property:

$$\mathcal{N} \models \phi \Leftrightarrow \mathcal{M} \models \phi .$$

So take any $\phi \in T$. Note that $FT(\phi) \subseteq \mathcal{T}$ and proceed by induction on the construction of ϕ. If ϕ is a quantifier formula then we have $\mathcal{N} \models \phi$ iff $\phi \in Q$ iff $\phi \in \mathcal{M}$ iff $\mathcal{M} \models \phi$.

If $\phi \equiv \tau_1 = \tau_2$ then we must have $\mathcal{T} \neq \emptyset$ and so $\mathcal{N} \models \tau_1 = \tau_2$ iff $\tau_1^{\mathcal{N}} = \tau_2^{\mathcal{N}}$ iff, by (3), $r(\tau_1) = r(\tau_2)$ iff, by (1), $\mathcal{M} \models \tau_1 = \tau_2$.

If $\phi \equiv P(\tau_1, \ldots, \tau_n)$ then we must have $\mathcal{T} \neq \emptyset$ and so $\mathcal{N} \models P(\tau_1, \ldots, \tau_n)$ iff $\langle \tau_1^{\mathcal{N}}, \ldots, \tau_n^{\mathcal{N}} \rangle \in P^{\mathcal{B}}$ iff, by (3), $\langle r(\tau_1), \ldots, r(\tau_n) \rangle \in P^{\mathcal{B}}$ iff, by (2), $\mathcal{M} \models P(\tau_1, \ldots, \tau_n)$.

If $\phi \equiv \phi_1 \vee \phi_2$ then $\mathcal{N} \models \phi_1 \vee \phi_2$ iff $\mathcal{N} \models \phi_1$ or $\mathcal{N} \models \phi_2$ iff, by IH, $\mathcal{M} \models \phi_1$ or $\mathcal{M} \models \phi_2$ iff $\mathcal{M} \models \phi_1 \vee \phi_2$. The remaining cases are similar. □

6.10 Quasitautological consequence. Fix a language $\mathcal{L}$ and let T be a set of formulas from $\mathcal{L}$. We define the relation ϕ *is a quasitautological consequence of* T, in symbols $T \models_i \phi$, as follows:

$$T \models_i \phi \Leftrightarrow (\mathcal{M} \models T \Rightarrow \mathcal{M} \models \phi) \text{ for all identity interpretations } \mathcal{M}.$$

We write $\models_i \phi$ for $\emptyset \models_i \phi$ and such a ϕ is *quasitautology*. Note that we have

$$\models_i \phi \Leftrightarrow \mathcal{M} \models \phi \text{ for all identity interpretations } \mathcal{M}.$$

6.11 Lemma. $\models_i Eq$.

Proof. Take any identity interpretation $\mathcal{M} = \langle Q, \mathcal{A}, a \rangle$ for $\mathcal{L}$. We wish to prove $\mathcal{M} \models \phi$ for an equality axiom ϕ. If ϕ is a predicate substitution axiom 6.8(5) then $\mathcal{M} \models \phi$ iff from $\mathcal{M} \models \tau_1 = \rho_1$, $\ldots$, $\mathcal{M} \models \tau_n = \rho_n$, and $\mathcal{M} \models P(\tau_1, \ldots, \tau_n)$ follows $\mathcal{M} \models P(\rho_1, \ldots, \rho_n)$. If the assumptions are satisfied we have $\tau_1^{\mathcal{M}} = \rho_1^{\mathcal{M}}$, $\ldots$, $\tau_n^{\mathcal{M}} = \rho_n^{\mathcal{M}}$, and $\langle \tau_1^{\mathcal{M}}, \ldots, \tau_n^{\mathcal{M}} \rangle \in P^{\mathcal{A}}$ and so $\langle \rho_1^{\mathcal{M}}, \ldots, \rho_n^{\mathcal{M}} \rangle \in P^{\mathcal{A}}$, i.e. $\mathcal{N} \models P(\rho_1, \ldots, \rho_n)$.

The remaining cases are similar. □

6.12 Theorem (Reduction of quasitautological consequence to tautological).

$$T \models_i \phi \Rightarrow T, Eq^{T \cup \{\phi\}} \models_p \phi \tag{1}$$

$$T \models_i \phi \Leftrightarrow T, Eq \models_p \phi \ . \tag{2}$$

Proof. (1): Assume $T \models_i \phi$ and take any propositional interpretation $\mathcal{M}$ such that $\mathcal{M} \models T \cup Eq^{T \cup \{\phi\}}$ holds. From Lemma 6.9 we obtain a $T \cup \{\phi\}$-equivalent identity interpretation $\mathcal{N}$ and so $\mathcal{N} \models T$. We then get $\mathcal{N} \models \phi$ from the assumption and $\mathcal{M} \models \phi$ from the equivalence.

(2): The direction ($\Rightarrow$) follows from (1) by weakening. In the direction ($\Leftarrow$) assume $T, Eq \models_p \phi$ and take any identity interpretation $\mathcal{M}$ for $\mathcal{L}$ such that $\mathcal{M} \models T$ holds. We have $\mathcal{M} \models Eq$ by Lemma 6.11. From Lemma 6.7 we obtain an equivalent propositional interpretation $\mathcal{N}$. We thus have $\mathcal{N} \models T \cup Eq$, we get $\mathcal{N} \models \phi$ from the assumption, and $\mathcal{M} \models \phi$ from the equivalence. □

6.13 Decidability of quasitautologies. One of the consequences of the Reduction theorem is that the predicate $\models_i \phi$ of being a quasitautology is decidable. We namely have

$$\models_i \phi \Leftrightarrow \models_p \bigwedge Eq^{\{\phi\}} \to \phi \ . \tag{1}$$

First note that the set $FT(\phi)$ is finite and so there are only finitely many equality axioms with free terms from $FT(\phi)$ which means that the set $Eq^{\{\phi\}}$ is finite. We have $\models_i \phi$ iff, by 6.12(2), $Eq \models_p \phi$ iff, by Semantic deduction lemma 5.2, $\models_p \bigwedge Eq^{\{\phi\}} \to \phi$.

6.14 Theorem. $T \models_p \phi \Rightarrow T \models_i \phi$.

Proof. If $T \models_p \phi$ holds then we have $T, Eq \models_p \phi$ by weakening and $T \models_i \phi$ by 6.12(2). □

6.15 Theorem. *To every identity interpretation $\mathcal{M}$ for $\mathcal{L}$ there is an equivalent canonical identity interpretation $\mathcal{N}$.*

Proof. For a given identity interpretation $\mathcal{M}$ for $\mathcal{L}$ we obtain an equivalent propositional interpretation $\mathcal{M}_1$ by Lemma 6.7. By Lemma 6.11 we have $\mathcal{M} \models Eq$ and so $\mathcal{M}_1 \models Eq$ by the equivalence. For the set T consisting of all formulas we have $Eq = Eq^T$ and we obtain a T-equivalent, i.e. equivalent, canonical identity interpretation $\mathcal{N}$ by Lemma 6.9. □

Syntax

6.16 Identity tableau expansion rules. Fix a language $\mathcal{L}$. All expansion rules for identity are unary with premises in assumptions. In the following $\tau, \tau_1, \ldots, \tau_n, \rho_1, \ldots, \rho_n$ are arbitrary terms. *Reflexivity, symmetry,* and *transitivity* rules are in that order:

$$\frac{}{\tau = \tau}\,(\mathit{Refl}) \qquad \frac{\tau_1 = \tau_2}{\tau_2 = \tau_1}\,(\mathit{Sym}) \qquad \frac{\tau_1 = \tau_2 \ \ \tau_2 = \tau_3}{\tau_1 = \tau_3}\,(\mathit{Trans})$$

Function substitution rules are for every n-ary function symbol f of $\mathcal{L}$ with $n > 0$ as follows:

$$\frac{\tau_1 = \rho_1 \quad \cdots \quad \tau_n = \rho_n}{f(\tau_1, \ldots, \tau_n) = f(\rho_1, \ldots, \rho_n)}\,(\mathit{Fsub})$$

Predicate substitution rules are for every n-ary predicate symbol P of $\mathcal{L}$ with $n > 0$ as follows:

$$\frac{\tau_1 = \rho_1 \quad \cdots \quad \tau_n = \rho_n \quad P(\tau_1, \ldots, \tau_n)}{P(\rho_1, \ldots, \rho_n)}\,(\mathit{Psub})$$

We can see that every identity rule *corresponds* to exactly one equality axiom.

6.17 Proofs with identity tableaux. An identity tableau π for a sequence of signed formulas Δ from axioms in T is called *identity tableau for Δ from axioms T*. If π is closed then we assert this by writing $\pi : T \vdash_i [\Delta]$. When $\Delta \equiv \phi*$ then we say that π *is identity tableau proving ϕ from axioms T* and write it as $\pi : T \vdash_i \phi$. We use additional abbreviations similar to those discussed in Par. 3.11.

6.18 Theorem (Admissible expansion rules in identity tableaux). *All rules proved admissible for propositional tableaux in Sect. 4 are also admissible in identity tableaux.*

Proof. Inspection of the proofs of admissibility of expansion rules in Sect. 4 reveals that the proofs remain correct also for identity tableaux, the presence of identity rules does not affect the proofs.

The only potential problem could be in the admissibility of cuts on propositional atoms (see Lemma 4.8) because we now have identity rules on propositional atoms $\tau_1 = \tau_2$ in assumptions. Fortunately, the elimination of cuts on such formulas removes the goals $\tau_1 = \tau_2*$ which are not affected by the identity expansion rules. □

6.19 Lemma. $\vdash_i Eq$.

Proof. An axiom of reflexivity 6.8(1) has the following proof:

$$\frac{\tau = \tau*}{\begin{array}{ll}\tau = \tau & (Refl)\\ \circ & \end{array}}$$

An axiom of predicate substitution 6.8(5) has the following proof:

$$\frac{\tau_1 = \rho_1 \wedge \cdots \wedge \tau_n = \rho_n \wedge P(\tau_1, \ldots, \tau_n) \rightarrow P(\rho_1, \ldots, \rho_n)*}{\begin{array}{ll} P(\rho_1, \ldots, \rho_n)* & (\rightarrow_2 *)\\ \tau_1 = \rho_1 \wedge \cdots \wedge \tau_n = \rho_n \wedge P(\tau_1, \ldots, \tau_n) & (\rightarrow_2 *)\\ \tau_1 = \rho_1 & (G\wedge_1)\\ \cdots & \\ \tau_n = \rho_n & (G\wedge_n)\\ P(\tau_1, \ldots, \tau_n) & (G\wedge_{n+1})\\ P(\rho_1, \ldots, \rho_n) & (Psub)\\ \circ & \end{array}}$$

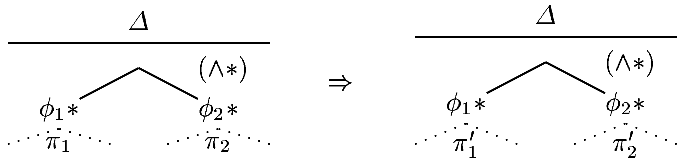
Remaining axioms have similar proofs. □

6.20 Lemma (Introduction of identity rules).

$$\pi : T, Eq \vdash_p [\Delta] \Rightarrow T \vdash_i [\Delta]$$

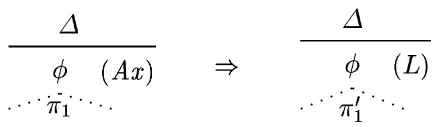
Proof. By induction on the structure of the tableau π. Assume $\pi : T, Eq \vdash_p [\Delta]$ and consider the form of π. If π is empty then we trivially have $T \vdash_i [\Delta]$.

If the first expansion in π is by a propositional rule, say $(\wedge *)$, then $\phi_1 \wedge \phi_2 * \in \Delta$ and π has a form shown in the following on the left:



We obtain identity tableaux π_1' and π_2' such that $\pi_1' : T \vdash_i [\Delta, \phi_1 *]$ and $\pi_2' : T \vdash_i [\Delta, \phi_2 *]$ by two IH's and construct the closed identity tableau for Δ from axioms T shown on the right. The remaining expansions by propositional rules and by axiom rules from T are similar.

If the first expansion in π is by an axiom rule for $\phi \in Eq$ then π has a form shown in the following on the left:



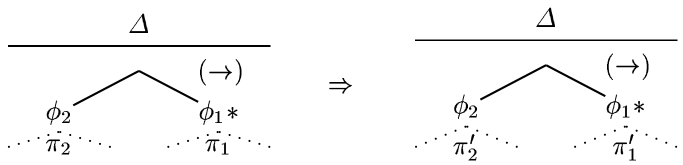
We obtain an identity tableau π_1' such that $\pi_1' : T \vdash_i [\Delta, \phi]$ by IH, note that $\vdash_i \phi$ by Lemma 6.19, and construct by a lemma rule the closed identity tableau for Δ from axioms T shown on the right. □

6.21 Lemma (Elimination of identity rules).

$$\pi : T \vdash_i [\Delta] \Rightarrow T, Eq \vdash_p [\Delta] \ .$$

Proof. By induction on the structure of the tableau π. Assume $\pi : T \vdash_i [\Delta]$ and consider the form of π. If π is empty then we trivially have $T \vdash_p [\Delta]$ and we obtain $T, Eq \vdash_p [\Delta]$ by weakening.

If the first expansion in π is by a propositional rule, say $(\rightarrow)$, then $\phi_1 \rightarrow \phi_2 \in \Delta$ and π has a form shown in the following on the left:



We obtain propositional tableaux π_1' and π_2' such that $\pi_1' : T, Eq \vdash_p [\Delta, \phi_1 *]$ and $\pi_2' : T, Eq \vdash_p [\Delta, \phi_2]$ by two IH's and construct the closed propositional tableau for Δ from axioms T, Eq shown on the right. The remaining propositional and axiom expansions are similar.

If the first expansion in π is by a reflexivity rule then π has a form shown in the following on the left:

$$\frac{\Delta}{\tau = \tau \quad (\mathit{Refl})} \qquad \Rightarrow \qquad \frac{\Delta}{\tau = \tau \quad (Ax)}$$
$$\cdots\bar{\pi}_1\cdots \qquad\qquad\qquad \cdots\bar{\pi}'_1\cdots$$

We obtain a propositional tableau π'_1 such that $\pi'_1 : T, Eq \vdash_p [\Delta, \tau = \tau]$ by IH and construct the closed propositional tableau for Δ from axioms T, Eq shown on the right.

If the first expansion in π is by a function substitution rule then π has a following form:

$$\frac{\Delta}{f(\tau_1, \ldots, \tau_n) = f(\rho_1, \ldots, \rho_n) \quad (\mathit{Fsub})} \quad \text{where } \tau_1 = \rho_1, \ldots, \tau_n = \rho_n \in \Delta$$
$$\cdots\bar{\pi}_1\cdots$$

We obtain a propositional tableau π'_1 such that

$$\pi'_1 : T, Eq \vdash_p [\Delta, f(\tau_1, \ldots, \tau_n) = f(\rho_1, \ldots, \rho_n)]$$

by IH and construct the following closed propositional tableau for Δ from axioms T, Eq:

$$\frac{\Delta \quad \text{s.t. } \tau_1 = \rho_1, \ldots, \tau_n = \rho_n \in \Delta}{\tau_1 = \rho_1 \land \cdots \land \tau_n = \rho_n \to f(\tau_1, \ldots, \tau_n) = f(\rho_1, \ldots, \rho_n) \quad (Ax)}$$
$$(\to)$$
$$\begin{array}{cc} f(\tau_1, \ldots, \tau_n) = f(\rho_1, \ldots, \rho_n) & \tau_1 = \rho_1 \land \cdots \land \tau_n = \rho_n * \quad (G\land *) \\ \cdots\bar{\pi}'_1\cdots & \begin{array}{ccc} \tau_1 = \rho_1 * & \cdots & \tau_n = \rho_n * \\ \circ & & \circ \end{array} \end{array}$$

The remaining identity expansion rules are similar. □

6.22 Theorem (Reduction of identity tableaux to propositional tableaux).

$$T \vdash_i \phi \Leftrightarrow T, Eq \vdash_p \phi \ .$$

Proof. In the direction ($\Rightarrow$) assume $\pi : T \vdash_i \phi$, i.e $\pi : T \vdash_i [\phi *]$, and obtain $T, Eq \vdash_p [\phi *]$, i.e. $T, Eq \vdash_p \phi$, by Lemma 6.21.

In the direction ($\Leftarrow$) assume $T, Eq \vdash_p \phi$, i.e. $T, Eq \vdash_p [\phi *]$, and obtain $T \vdash_i [\phi *]$, i.e. $T \vdash_i \phi$, by Lemma 6.20. □

6.23 Corollary (Soundness and completeness of identity tableaux). *For any fromula ϕ and a set of formulas T from $\mathcal{L}$ we have*

$$\begin{array}{ccc} T \models_i \phi & & T \vdash_i \phi \\ \Uparrow\!\Downarrow 6.12(2) & & \Uparrow\!\Downarrow 6.22 \\ T, Eq \models_p \phi & \Leftrightarrow 5.13 & T, Eq \vdash_p \phi \end{array} \qquad \Box$$

6.24 Decidability of quasitautologies revisited. Property 6.13(1) reduces the problem of deciding whether a formula ϕ of some $\mathcal{L}$ is quasitautology to the problem of deciding whether the formula $\bigwedge Eq^{\{\phi\}} \to \phi$ is a tautology. This decision procedure can be called the method of *associates* because the conjunction of equality axioms $\bigwedge Eq^{\{\phi\}}$ is an *identity associate* of ϕ. Quantifier associates were introduced by R. Smullyan.

Lemma 6.7 and property 6.12(1) give us another method of deciding quasitautologies: test $\mathcal{N} \models \phi$ in all finite identity interpretations $\mathcal{N}$ for a finite language consisting of function and predicate symbols in ϕ. Note that the quantifier sets Q of such interpretations can be restricted to subsets of finitely many quantifier formulas among $FPA(\phi)$. Also note that there are only finitely many assignments a to be considered because the only values which matter are the finitely many values $a(i)$ where $v_i \in FT(\phi)$. The formula ϕ is a quasitautology iff $\mathcal{N} \models \phi$ for all such finite identity interpretations $\mathcal{N}$. Such a test for quasitautologies seems to be much less convenient than the method of associates. We mention it here because this is the method used by Boolos and Jeffrey to recognize proofs in their proof calculus for predicate logic.

Identity tableaux offer the most convenient test for quasitautologies. We call a branch Δ of an identity tableau *identically complete* if whenever an identity rule has its premises Δ_1 in Δ then also its conclusion ϕ is an assumption in Δ provided $FT(\phi) \subseteq FT(\Delta_1)$. The reader will note that the last condition can be violated only by function substitution rules whose conclusions can introduce new terms into identity tableaux.

In order to test whether ϕ is a quasitautology we construct a tableau for $\phi*$ and in every open propositionally complete branch we saturate the branch by the conclusions of finitely many identity rules which can be applied on the branch and which do not introduce new terms. If all branches close then ϕ is a quasitautology by Corollary 6.23. If there is an open propositionally and identically complete branch then we construct a propositional interpretation $\mathcal{M}$ by collecting all propositional atoms in the assumptions of the branch. We have proved in Par. 3.15 that we have $\mathcal{M} \not\models \phi$ and $\mathcal{M} \models \psi$ for all assumptions ψ in the branch. Now, the branch is identically complete, and so we must have $\mathcal{M} \models Eq^{\{\phi\}}$. But this means that $Eq^{\{\phi\}} \not\models_p \phi$ holds and so we have $\not\models_i \phi$ by 6.12(1).

Identity tableaux facilitate also a semidecidable test for quasitautological consequence from decidable axioms T: $T \models_i \phi$. This is an extension of the procedure described in Par. 5.14 where we identically complete every open branch which is propositionally complete just before we add the next axiom to the assumptions of the branch. By the same reasoning as above we can show that if there is an open branch which is propositionally, identically, and axiomatically complete then we have a propositional interpretation $\mathcal{M}$ such that $\mathcal{M} \models T \cup Eq^{T\cup\{\phi\}}$ and $\mathcal{M} \not\models \phi$. Hence $\mathcal{M}$ witnesses $T, Eq^{T\cup\{\phi\}} \not\models_p \phi$ and we get $T \not\models_i \phi$ by 6.12(1).