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A survey on interval routing

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Abstract

We survey in this paper the classical results, and also the most recent results, in the field of Interval Routing, a well-known strategy to code in a compact way distributed routing algorithms. These results are classified in several themes: characterization, compactness and shortest path, dilation and stretch factor, specific class of graphs (interconnection networks, bounded degree, planar, chordal rings, random graphs, etc.), and the other recent extensions proposed in the literature: dead-lock free, congestion, non-deterministic, and distributed problems related to Interval Routing. For each of these themes we review the state of the art and propose several open problems. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

Interval Routing is a way of implementing routing schemes on arbitrary networks. It is based on representing the routing table stored at each node in a compact manner, by grouping the set of destination addresses that use the same output port into intervals of consecutive addresses. A possible way to represent such a scheme is to use a connected undirected labeled graph, providing the underlying topology of the network. The addresses are assigned to the nodes, and the sets of destination addresses are assigned to each endpoint of the edges. The routing is computed in a distributed way with the following algorithm: at each intermediate node x, the routing process ends if the header y corresponds to x, otherwise it is forwarded with the message through an edge labeled by a set I such that $y \in I$.

As originally introduced in [67], the scheme required each set of destinations to consist of a single interval. So, the message is forwarded through the edge labeled by an interval $[a,b]$ such that $a \leqslant y \leqslant b$. The main advantage of this scheme is the low memory requirement to store the routing: $O(d)$ integers (the boundaries of the

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intervals) for each node of degree d. Early works appear also in [76], and the first article to be published was [68]. This scheme was subsequently generalized in [78] to more than one interval per edge. The Interval Routing method is implemented in the INMOS C104 router [59], and in the RCube router [83]. A performance study (by simulation) of the C104 router is given in [47].

A short survey has already been presented in [79]. Since then, a lot of work in the field has been done.

The paper is organized as follows:

- Section 2 introduces the formal definition of Interval Routing, and its different related notions: compactness, dilation, stretch factor, and some basic characterizations.
- Section 3 presents a large collection of results concerning specific classes of networks: interconnection networks, chordal rings, planar, treewidth bounded, random graphs, and other classes of graphs and graph operators. A summary ends the section.
- Section 4 presents several extensions of the standard model of Interval Routing: non uniform cost link models, and the hierarchical results, deadlock-free, non-deterministic, and congestion of Interval Routing, and also the recent use of Interval Routing for solving several distributed tasks.

Each paragraph ends with some open questions and conjectures. Several new results are added: the coding of Interval Routing (Section 2.3, Theorem 1 and Theorem 2), a lower bound concerning the treewidth (Section 3.5, Theorem 34), and a lower bound for K_n under the dynamic link-cost model which implies some hierarchical results of Interval Routing classes, (Section 4.1, Theorem 47 and Corollary 1).

2. The model of interval routing

This section presents the basic material for definitions and notations concerning Interval Routing.

2.1. *Definition of interval routing*

The underlying topology is a symmetric digraph $G=(V,E)$, V being the set of routers of the network, and each bidirectional link between the nodes x and y is represented by two opposite arcs of E: (x,y) and (y,x). We denote by $\deg(x)$ the number of neighbors of x. For the routing problems, we will also assume that the graph is always finite and connected; moreover it has no loops and no multi-arcs. Although in the common model, the graph is undirected, Interval Routing can be applied to general digraphs. The definition quoted below holds for weighted graphs. The distinction between weighted and unweighted graphs will be relevant later in Section 2.2 when the length of the routing paths induced will be taken into consideration.

Definition 1. Let $G=(V,E)$ be a graph. A pair $(\mathscr{L},\mathscr{I})$ is an *Interval Routing Scheme* on G (IRS for short), if the following conditions are satisfied:

1. $\mathscr{L}$ is a one-to-one labeling of V, $\mathscr{L}: V \to \{1,\dots,|V|\}$;

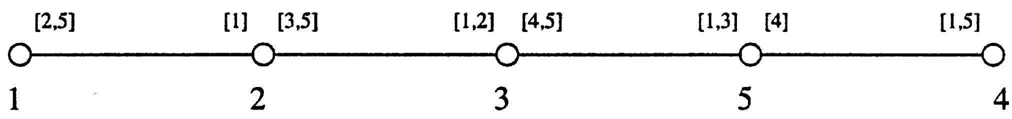

Fig. 1. A valid interval labeling scheme for a path.

2. $\mathscr{I}$ is an arc-labeling, $\mathscr{I}: E \to 2^{\mathscr{L}(V)}$, such that:[1]
   2a. for every $x \in V$, $\{\mathscr{I}(x,y) \mid (x,y) \in E\} \cup \mathscr{L}(x) = \{1,\dots,|V|\}$;
   2b. for every distinct arcs (x,y) and (x,z) of E, $\mathscr{I}(x,y) \cap \mathscr{I}(x,z) = \emptyset$;
3. for every $x, y \in V$, $x \neq y$, there exists a sequence of nodes $(u_0,\dots,u_t)$ such that $u_0 = x$, $u_t = y$, and for every $i \in \{1,\dots,t\}$, $\mathscr{L}(y) \in \mathscr{I}(u_{i-1},u_i)$.

Sometimes the set $\{0,\dots,|V|-1\}$ is used for the definition of the node-labeling $\mathscr{L}$. Note that any ordered set can be used as well.

An IRS on G induces a *routing function* on G, which is a function that returns for every source-destination pair (x,y) a path from x to y defined by the sequence of nodes of Condition 3. Such a path is called a *routing path*.

When a pair $(\mathscr{L},\mathscr{I})$ of labeling satisfies Conditions 1 and 2, it is called an Interval Labeling Scheme (ILS for short). An ILS on G is not necessary an IRS on G. In particular, Condition 3 fails if there exists a z, for some edge $\{x,y\}$, such that $z \neq x$, $z \neq y$ and $z \in \mathscr{I}(x,y) \cap \mathscr{I}(y,x)$. In this case we have an infinite loop between x and y whenever the destination of the message is z. Note that some label $\mathscr{I}(x,y)$ may be empty, and the link (x,y) not used. An IRS on a graph is a *valid* ILS, that is an ILS satisfying Condition 3. For the routing problem, we are only interested in valid ILS. The validity of every ILS (Condition 3) can be checked in $O(n^2)$ time [77], where n is the number of nodes of the graph.

We can check that Condition 1–3 for the example of Fig. 1 are satisfied. Of course many different labelings are possible for the same graph.

2.2. Compactness, linearity and strictness of interval routing

The notion of compactness and linearity are related to the way of coding the labels of the arcs, whereas strictness is related to the construction of the IRS itself. For integers n, $a,b \leqslant n$, an *interval* $[a,b]$ with respect to n is the set of consecutive integers between a and b, n and 1 being considered as consecutive. Formally, $[a,b] = \{i \mid a \leqslant i \leqslant b\}$, if $a \leqslant b$, and $[a,b] = \{i \mid b \leqslant i \leqslant n\} \cup \{i \mid 1 \leqslant i \leqslant a\}$ otherwise. These two kinds of intervals are respectively called *linear* and *cyclic* intervals. For every set $I \subseteq \{1,\dots,n\}$, the *number of intervals* of I is the length, k, of the smallest sequence $(a_1,b_1,\dots,a_k,b_k)$ such that $I = [a_1,b_1] \cup \cdots \cup [a_k,b_k]$. The number of linear intervals of I is defined similarly, but all intervals must be linear. The number of intervals of the empty set is 0.

[1] $2^{\mathscr{L}(V)}$ denotes the power set of $\mathscr{L}(V)$, i.e., the set of all the subsets of $\{1,\dots,|V|\}$.

Definition 2. Let $R=(\mathscr{L},\mathscr{I})$ be an IRS on a graph G. The *compactness* of R is the maximum, over all the arcs (x,y) of G, of the number of intervals of $\mathscr{I}(x,y)$.

The linear compactness is defined similarly from the number of linear intervals of the arc-labels. Compactness and linear compactness differ by at most 1: every cyclic interval equals two linear intervals. We denote by k-IRS every IRS of compactness k. A k-IRS that has also a linear compactness k is called a k-Linear Interval Routing Scheme (k-LIRS for short). Moreover, an IRS is qualified as *strict*, denoted by SIRS, if every arc (x,y) satisfies $\mathscr{L}(x)\notin\mathscr{I}(x,y)$. Hence we have four variants for IRS depending on linearity and strictness: k-IRS, k-LIRS, k-SIRS, and k-SLIRS. Section 4.2 gives more details about the hierarchy of the classes induced by these different IRS.

The IRS depicted in Fig. 1 is linear, non-strict, and of linear compactness 1. So this is a 1-LIRS.

2.3. Coding of interval routing

The routing information of a node is entirely determined by the knowledge of all the labels (its local name and the labels of its incident arcs). Implicitly, in the Interval Routing model one can permute the output ports numbers in advance. We mean that for the routing decision of a node x towards a destination y, x is able to determine the output port number such that $\mathscr{L}(y)\in\mathscr{I}(e)$, and to send the message onto this port (i.e., through the edge e) without extra information, excepted, of course, the knowledge of the label $\mathscr{I}(e)$.

We invite the reader to see [7] for a discussion of the impact of the node and/or port relabeling on routing information complexity. In all the following, the function log denotes the logarithm in base 2.

Theorem 1. *Every k-IRS (and its variants) on an n-node graph can be implemented in each node x with $\log\binom{n}{K}+\log\binom{K}{d}+(K-d)\log d+\mathrm{O}(\log n)$ bits, which is in $\mathrm{O}(dk\log(n/k))$ bits, where K is the total number of intervals for the node x, and d the number of arcs incident to x that have a non-empty label.*

Proof. Let x be a node. For each $i\in\{1,\dots,K\}$, we denote by $[a_i,b_i]$ the ith interval of x (with Condition 2b the intervals do not overlap), and by e_i the output port number on which is assigned this interval. First, we remark that only 2 intervals at most can be non-strict and/or cyclic. So, with an overhead of $\mathrm{O}(\log n)$ bits only, it is easy to implement any k-IRS, k-SIRS, or k-LIRS, from its strict and linear version.

W.l.o.g. we assume that the a_i's are sorted in increasing order, the intervals $[a_i,b_i]$ are strict and linear, and we know the integers n, d, K, and $\mathscr{L}(x)$ with an overhead of $5\log n$ bits.[2] To code all the labels in x, it suffices to store the sequences $S_1=(a_1,\dots,a_K)$ and $S_2=(e_1,\dots,e_K)$. Indeed the b_i's can be computed as follows: If

[2] $4\log n+\mathrm{O}(\log\log n)$ bits suffice to store n, d, K, $\mathscr{L}(x)$ in a self-delimiting way, which is bounded by $5\log n$ for n large enough.

$i < K$, then $b_i = a_{i+1} - 1$ ($= a_{i+1} - 2$ if $\mathscr{L}(x) = a_{i+1} - 1$), otherwise $b_K = n$ ($= n - 1$ if $\mathscr{L}(x) = n$). S_1 is a sequence of K distinct integers in the range 1 to n, therefore it can be stored with $\log\binom{n}{K}$ bits [56].

Let d be the number of output ports used by the IRS in x, i.e., $d = |\{e_i \mid 1 \leqslant i \leqslant K\}|$. The IRS does not use necessarily all the arcs incident to x, so d might be less than the degree of x. The number of ways to obtain S_2 is at most $\binom{K}{d} d^{K-d}$. Indeed, $S_2 = (e_1, \ldots, e_K)$ is a sequence composed of exactly d different values e_i taken from $\{1, \ldots, d\}$ located in K possible places, and of $K-d$ independent integers of $\{1, \ldots, d\}$. So, there is at most $d!\binom{K}{d}$ choices of the d different values of the sequences, and d^{K-d} ways for the $K-d$ other elements; thus a total of $d!\binom{K}{d} d^{K-d}$ different sequences $(e_1, \ldots, e_K)$. However the permutation of the output ports numbers can be made in advance. Since d ports are used, at least $d!$ sequences are equivalent up to a permutation of the ports, coding the same IRS. Each different IRS has a sequence S_2 which can be coded with $\log(d!\binom{K}{d} d^{K-d}/d!) = \log\binom{K}{d} + (K-d)\log d$ bits.

Thus the total number of bits used to store all the labels in x is bounded by $M = \log\binom{n}{K} + \log\binom{K}{d} + (K-d)\log d + c\log n$, for a suitable constant $c \leqslant 5$. On the other hand, $k \leqslant K \leqslant dk$. Moreover $\binom{n}{K} \leqslant (n\mathrm{e}/K)^K \leqslant (n\mathrm{e}/k)^{dk}$, and $\log\binom{K}{d} \leqslant K$. Hence

$$\log\binom{n}{K} + \log\binom{K}{d} \leqslant K\left(\log\frac{n}{k} + \log(2\mathrm{e})\right) \leqslant \log(2\mathrm{e})\, dk\log\frac{n}{k}.$$

We note that k is in the range $1 \leqslant k \leqslant n/2$, thus for n large enough, $(n/k)^k \geqslant n$, and therefore $c\log n \leqslant ck\log(n/k) \leqslant c\,dk\log(n/k)$, for every constant $c \geqslant 0$, and for every $d \geqslant 1$. It follows that

$$\log\binom{n}{K} + \log\binom{K}{d} + c\log n \leqslant (c + \log(2\mathrm{e}))dk\log\frac{n}{k}.$$

To prove that $M = \mathrm{O}(dk\log(n/k))$, it remains to show that

$$(K-d)\log d \leqslant \alpha\, dk\log\frac{n}{k}, \quad \text{for a suitable constant } \alpha \geqslant 1. \tag{1}$$

Let assume that $n > dk$. In this case:

$$d < \frac{n}{k} \Rightarrow \log d < \log\frac{n}{k} \Rightarrow K\log d < dk\log\frac{n}{k} \Rightarrow (K-d)\log d < dk\log\frac{n}{k}.$$

It remains to show Inequality 1 for $n \leqslant dk$. Let $\beta = dk/n$, $\beta \geqslant 1$. Since $K \leqslant n$, we get $K \leqslant dk/\beta$. To show Inequality 1 it suffices to show:

$$K\log d \leqslant \alpha\, dk\log\frac{d}{\beta}, \quad \text{or} \quad \frac{dk}{\beta}\log d \leqslant \alpha\, dk\log\frac{d}{\beta}$$

$$\Leftrightarrow \quad \log d \leqslant \alpha\beta\log\frac{d}{\beta}$$

$$\Leftrightarrow \quad d \leqslant \left(\frac{d}{\beta}\right)^{\alpha\beta}.$$

Let $f(\beta)=(d/\beta)^{\alpha\beta}$. Thus, $f(\beta)$ increases if $f'(\beta)\geqslant 0$ for $\beta\geqslant 1$.

$$f'(\beta)=\alpha f(\beta)\left(\ln\left(\frac{d}{\beta}\right)-1\right).$$

$\alpha f(\beta)>0$. Thus $f'(\beta)\geqslant 0$ if $\ln(d/\beta)\geqslant 1$, or $d\geqslant\beta\mathrm{e}$. So, if $d\geqslant\beta\mathrm{e}$ then $f(\beta)\geqslant f(1)=d^{\alpha}$. Therefore if $d\geqslant\beta\mathrm{e}$ then Inequality 1 holds.

So, let us assume $d<\beta\mathrm{e}$. If, furthermore, $\beta\mathrm{e}\leqslant\gamma\log d$, where $\gamma=3/\log 3\approx 1.892$, then it implies that $d<\gamma\log d$, which is impossible for $d\geqslant 1$. So, $\beta\mathrm{e}>\gamma\log d$. In this case it implies that

$$\begin{aligned} n\beta e>\gamma n\log d &\Rightarrow dk\mathrm{e}>\gamma n\log d,\ \text{because } dk=n\beta\\ &\Rightarrow dk\mathrm{e}>\gamma(K-d)\log d,\ \text{because } n>K-d\\ &\Rightarrow (K-d)\log d<\frac{\mathrm{e}}{\gamma}\,dk\log\frac{n}{k}. \end{aligned}$$

In total,

$$M<\left(c+\log(2\mathrm{e})+\frac{\mathrm{e}}{\gamma}\right)dk\log\frac{n}{k}<9\,dk\log\frac{n}{k}. \qquad \square$$

Theorem 1 implies that every 1-IRS (and its variants) can be coded with $n+\mathrm{O}(\log n)$ bits per node. Moreover, the implementation is quite easy using an n-bit vector coding within the 1's the left-boundary of each interval. The time complexity of the routing function is linear in n in this case (the time to locally compute the output port from any destination). However, adding a table of $\lceil n/f(n)\rceil$ integers, i.e., $\mathrm{o}(n)$ extra bits for some function f such that $\log n=\mathrm{o}(f(n))$, the routing function can compute the output port in $\mathrm{O}(f(n))$ bit-operations: Split the vector in $\lceil n/f(n)\rceil$ blocks of length at most $f(n)$ bits, and tabulate for the ith block the number of 1's which is contained in the vector up to the position $i\cdot f(n)$.

One can even reduce the amount of bits needed to route for trees.

Theorem 2. *Every n-node tree has a 1-SIRS which can be implemented with $\mathrm{O}(\sqrt{n})$ bits in each node.*

Proof. Let T be an n-node tree, and r be a node of T chosen as the root of T. For every edge (u,v), the graph obtained by removing (u,v) in T is composed of two connected components. We denote by $T_{(u,v)}$ the component that contains v. We say that $T_{(u,v)}$ is the subtree of T induced by (u,v). For all integers n,k, a k-partition of n is an integer sequence $(n_1,\dots,n_k)$ such that $1\leqslant n_1\leqslant\cdots\leqslant n_k$, and $\sum_{i=1}^{k}n_i=n$.

We label the nodes of T with a particular depth first search scheme as follows: We initialize the labeling process by labeling r with 1. For each node x, let $y_1,\dots,y_k$ be the children of x ordered such that $|V(T_{(x,y_1)})|\leqslant\cdots\leqslant|V(T_{(x,y_k)})|$. We start to label recursively the subtrees $T_{(x,y_1)},\dots,T_{(x,y_k)}$ in this order. If $x\neq r$, we assign the output port number 1 to the edge towards r, i.e., the edge (x,y) with y the father of x, and for each $i \in \{1,\ldots,k\}$ the output port number $i+1$ to the edge (x, y_i). For $x=r$ we assign the output port number i to the edge (x, y_i).

Consider a node $x \neq r$. We set $k = \deg(x) - 1$, for each $i \in \{1,\ldots,k\}$, set $n_i = |V(T_{(x,y_i)})|$, and finally we set $n_0 = n - \sum_{i=1}^{k} n_i$. We remark that $\sum_{i=0}^{k} n_i = n$, or equivalently $(n_1,\ldots,n_k)$ is a k-partition of $n - n_0$. We store in x:

- the label of x;
- the values n, n_0, and k;
- the k-partition of $n-n_0$: $(n_1,\ldots,n_k)$.

For $x=r$ we set $k = \deg(x)$, and $n_0 = 1$, and we store similarly the values n, k, and the k-partition of $n - n_0$ defined by $(n_1,\ldots,n_k)$. To simplify, x and y represent the labels of the node x and y respectively.

The routing scheme is the following: Assume the node x must route a message to the destination y. If $x=y$ then the routing process ends. If $y \notin [x+1, x+n-n_0]$, then the message is forwarded to the father of x through the output port 1 (the case never happens if $x=r$). Otherwise, one computes the unique integer $p \geqslant 1$ such that $y \in [x+1+\sum_{i=1}^{p-1} n_i, x+\sum_{i=1}^{p} n_i]$. The message is forwarded through the output port $p+1$, if $x \neq r$, and p for $x=r$. Clearly such a scheme corresponds to a 1-SIRS, because the union of all the intervals cover $\{1,\ldots,n\}$, and all the intervals are pairwise disjoint. Moreover they never contain x.

The process routes correctly. Indeed, by the construction of the node-labeling if $y \in [x+1+\sum_{i=1}^{p-1} n_i, x+\sum_{i=1}^{p} n_i]$, then y is necessarily a node of the subtree $T_{(x,y_p)}$, and the output port assigned to the edge (x, y_p) is $p+1$ if $x \neq r$, and p if $x=r$. And if $y \notin [x+1, x+n-n_0]$, then y is not a descendent of x, and hence must be forwarded to its father.

Let us compute the amount of information required. The integer values of n, n_0, k, and the label of x can be stored using $O(\log n)$ bits. Knowing n and k, any k-partition of n, P, can be coded using at most $\lceil \log U_n \rceil$ bits, where U_n is the total number of partitions of n. Indeed, there exists very simple algorithm that, knowing n, enumerates all the partitions of n. So, it suffices to store the index of P in such an enumeration. Furthermore, we have the well-known formula due to Harder and Ramanujan in 1917 [45, Equation (4.2.7) p. 44]:

$$U_n \sim \frac{1}{4n\sqrt{3}}\, e^{\pi\sqrt{2n/3}}.$$

Globally $3.71\sqrt{n}$ bits per node suffice to describe the routing algorithm for n large enough, that completes the proof. $\square$

Note that in [12], it is shown that to route in an arbitrary n-node tree $\Omega(\sqrt{n})$ bits are required, showing that the bound of Theorem 2 is tight.

We remark that the distinction between the variants of Interval Routing has no real impact on the coding of IRS in local memory of the nodes, up to an additive term of $O(\log n)$ bits. The interest of these variants will appear in Theorem 43 of Section 3.8.

Since every edge needs $O(k \log n)$ bits of information to code its labels, the total amount of routing information for the entire graph $G=(V,E)$ is bounded by $O(|E|k \log n)$ bits.

2.4. Characterizations

Every graph supports an IRS: Label arbitrary the nodes and select any routing function with simple paths (without loop) to generate the label of the arcs. Because Interval Routing has been investigated to give compact implementation of routing functions, we are interested to find IRS with small compactness.

Using a depth first search for the labeling of nodes, we have the following result:

Theorem 3 (Santoro and Khatib [68]). *Every acyclic digraph has a* 1-*SIRS.*

Trees have a 1-SIRS (by Theorem 2, and also by Theorem 3 considering undirected trees as acyclic digraphs), and thus, by the use of a spanning tree, every graph has a 1-SIRS. This result has been extended:

Theorem 4 (van Leeuwen and Tan [78]). *Every graph has a* 1-*SIRS such that all the arcs have non empty labels.*

For the latter result, more details can be found in [71, p. 136]. A comparison of the labeling of [68] and of [78] is presented in [66]. However the result does not hold for linear intervals.

Theorem 5 (Fraigniaud and Gavoille [30])
- *A graph has a* 1-*LIRS if and only if it is not a lithium graph, i.e., a graph with three bridges connected to the same* 2-*edge-connected component, the bridges corresponding to edges of nodes of degree* 1 *excluded.*
- *A graph has a* 1-*SLIRS if and only if it is not a weak-lithium graph, i.e., a graph with three bridges connected to the same* 2-*edge-connected component.*

[30] derived an $O(n^2)$ time algorithm to label graphs supporting 1-LIRS and 1-SLIRS. A simpler algorithm, based on depth first search, is presented in [15]. (The time complexity of the latter is not mentioned.) Note that while theorems above provide characterizations nothing is said about the length of the routing paths.

2.5. Efficiency of interval routing: dilation and stretch factor

Whereas compactness, linearity, and strictness are qualitative parameters of the labeling, we can also define several other criteria to measure the quality of the routing induced by the IRS. Here the graphs considered have uniform weights on all the links.

Definition 3. Let R be an IRS on a graph G. The *dilation* of R, denoted by dilation(R), is the length of the longest routing path induced by R. The k-*dilation* of G, denoted by k-dilation(G), is the minimum, over all the k-IRS R on G, of the dilation of R.

By taking a labeling like a depth first search along a minimum spanning tree, the result of [68] implies that every graph G of diameter D satisfies 1-dilation$(G) \leqslant 2D$ (more precisely, twice the radius of G). Moreover, by combining Theorem 2, we can conclude that every graph of diameter D has a 1-SIRS of dilation at most $2D$ which can be implemented with $O(\sqrt{n})$ bits in each router.

First let us emphasize that the behavior between 1-IRS and 1-LIRS is different for the dilation.

Theorem 6 (Eilam et al. [13]). *For every fixed D, there exists a graph G of diameter at least D such that every 1-LIRS on G has a dilation at least $D^2/18$. Moreover G is planar and of maximum degree 4.*

The following result of [74] shows that the IRS proposed by [68] based on a spanning tree is close to the optimal.

Theorem 7 (Tse and Lau [74]). *For every even D, there is a graph G of diameter D, and of girth $2D$, such that* 1-dilation$(G) \geqslant 2D - 3$.

However, allowing more than one interval per edge, it is possible to decrease the dilation.

Theorem 8 (Královič et al. [51]). *For every n-node graph G of diameter D, there exists a $k \leqslant \lceil \sqrt{n \ln n} \rceil + 1$ such that* k-dilation$(G) \leqslant \lceil 3D/2 \rceil$.

Combined with the result of [40] (cf. the remark of Theorem 15), it is possible to improve the labeling of the regions defined in the proof of [51] in order to show that actually one can choose $k \leqslant \frac{1}{4}\sqrt{n \ln n} + O(n^{1/4} \log^{3/4} n)$.

Theorem 9 (Gavoille [37]). *For every $D \geqslant 2$, there exists an n-node graph G of diameter D such that* k-dilation$(G) \geqslant \lfloor 3D/2 \rfloor - 1$, *for every $k \leqslant cn/(D \log(n/D))$, and for a suitable constant c.*

It should be noted that the dilation is quite sensitive to the additive term around 3/2 the diameter.

For bounded degree graphs we have:

Theorem 10 (Flammini and Nardelli [23]). *For every $D \geqslant 4 \log n$ there exists an n-node bounded degree graph G of diameter D such that* k-dilation$(G) \geqslant 3D/2 - 2 \log (n/D)$, *for every $k \leqslant 0.05\, n/(D \log(n/D))$.*

Open Question 1

- Is the lower bound $\Omega(D^2)$ for the dilation of 1-LIRS tight?
- What is the smallest integer k (as function of n and D) such that k-dilation$(G) \leqslant \lceil 3D/2 \rceil$, for every n-node graph G of diameter D?

Although, using a 1-IRS based on depth first search, the routing path lengths remains always linear in the diameter, this length might be very large for nodes that are relatively close.

Definition 4. Let R be an IRS on a graph G. The *stretch factor* of R, denoted by stretch(R), is the smallest real s such that for every pair of nodes (x, y) the length of the routing path induced by R from x to y is at most s times longer than the distance in G between x and y. The *k-stretch* of G, denoted by k-stretch(G), is the minimum, over all the k-IRS R on G, of the stretch factor of R.

In practice we are interested in designing k-IRS with the smallest possible stretch factor. So, IRS of stretch factor 1 is of special interest.

Definition 5. Let R be an IRS on a graph G. R is a *shortest path* IRS if the routing paths induced by R on G are only composed of shortest paths in G.

So, shortest path IRS are IRS of stretch factor 1.

Definition 6. The *compactness* of a graph G, denoted by IRS(G), is the smallest integer k such that G supports a shortest path k-IRS.

The compactness of $G=(V,E)$ is called sometimes the IRS number of G. As we saw previously the space complexity for the storage of a routing scheme in x is $O(dk\log(n/k))$ bits or $O(|E|k\log n)$ bits in total. Therefore the compactness, k, is an important parameter in the design of compact routing schemes.

We can define similarly the linear compactness (LIRS(G)) and the strict or strict-linear compactness (SIRS(G) and SLIRS(G)) of graphs. However, up to an additive constant these variants are equivalent.

Definition 7. Let R be an IRS on a graph G, and $t \geqslant 0$ be an integer. R is a *t-regional* IRS on G if for every node x of G the induced routing paths by R from x to any node y at distance at most t of x are shortest paths.

The dilation of any t-regional IRS is at most $n-(t+1)$, if $n-(t+1) > t+1$, i.e., if $n \geqslant 2t+3$. So, the stretch factor of a t-regional IRS is at most $n/(t+1)-1$.

Theorem 11 (van Leeuwen and Tan [78])
- *Every n-node graph G has a* 1-*regional* 2-*SIRS. Hence, for every* $n \geqslant 5$, 2-stretch(G) $\leqslant n/2-1$.
- *Every Hamiltonian n-node graph H has a* 1-*regional* 1-*SIRS. Hence, for every* $n \geqslant 5$, 1-stretch(H) $\leqslant n/2-1$.

Open Question 2
- Is there an n-node graph G, for every n large enough, such that 1-stretch(G) $= \Omega(\sqrt{n})$? (The best known lower bound is a constant. It is derived from the result of [73] showing the construction of a graph G such that 1-stretch(G) $\geqslant 7$).

- What are the general bounds for k-stretch(G), $k>1$?
- Do all graphs support a 1-regional 1-IRS?

We will see that there is no hope to extend Theorem 11 to 2-regional IRS. Indeed in Paragraph 3.1, Theorem 16 states that shortest path IRS in n-node graphs of diameter 2 require $n/4 - \mathrm{o}(n)$ intervals in the worst-case. The 2-regional linear IRS satisfy the following property:

Theorem 12 (Zerrouk et al. [84]). *If G supports a 2-regional 1-SLIRS then every edge of G is either a bridge, or belongs to a cycle of length 3 or 4.*

The parameters dilation and stretch factor do not necessary reflect the quality of a routing scheme. The density of long routing paths may be small. We believe that for a better analysis of the real networks it would be relevant to consider the average dilation and the average stretch factor.

In the next definition $\rho(x,y)$ denotes the routing path length induced by the routing scheme R from x to y, and $s(x,y)$ the ratio between $\rho(x,y)$ and the distance between x and y.

Definition 8. Let R be an IRS on an n-node graph G. The *average dilation* (respectively *average stretch factor*) of R, is the real

$$\frac{1}{n(n-1)}\sum_{x\neq y}\rho(x,y) \quad \text{respectively,} \quad \frac{1}{n(n-1)}\sum_{x\neq y} s(x,y).$$

The average k-dilation(G) (respectively average k-stretch(G)) is the minimum, overall the shortest path k-IRS R on G, of the average dilation of R (respectively stretch factor).

The average stretch factor has been studied for Interval Routing in 2D-grids in [66]. For general graphs we have:

Theorem 13 (Eilam et al. [12]). *For every n-node weighted graph G of diameter D, there exists a k-SIRS on G (polynomial time constructible), $\leqslant 3\sqrt{n(1+\ln n)}$, of stretch factor at most 5, average stretch factor less than 3, and dilation $2D$. Moreover, the dilation is at most $\lceil 3D/2 \rceil$ if all the weights are uniform.*

Open Question 3

- What are the bounds on average dilation and average stretch factor of k-IRS for arbitrary graphs?

2.6. Complexity of characterization

The characterization of IRS with at most 1 interval per arc (linear or cyclic) is polynomial if there is no constraint about the path lengths induced by the routing, see Section 2.4. The problem becomes much harder when, for example, the shortest paths are required. Because the labeling can be optimized over all the possible node- and arc-labeling (including all the possible choice of routing paths), it appears that determining whether or not a graph has a shortest path 1-IRS is difficult in practice. The best-known NP-completeness results are the following:

Theorem 14.
- (*Eilam et al.* [14]) *For every s*, $1 \leqslant s < 3/2$, *the problem of determining whether a graph supports a* 1-*SLIRS* (*or its variants*) *of stretch factor s is NP-complete.*
- (*Flammini* [18]) *For every s*, $1 \leqslant s < 3/2$, *and for every integer* $k \geqslant 2$, *the problem of determining whether a graph supports a k-SLIRS* (*or its variants*) *of stretch factor s is NP-complete.*

Thus it is NP-complete for the case of weighted graphs (NP-completeness results are stated for the unweighted case), and shortest path IRS ($s = 1$). The result concerning the stretch factor was not mentioned in the original paper of [18]. However, since their construction is a graph of diameter 2, is quite easy to see that G has a shortest path 1-SLIRS (or its variant) if and only if G has a 1-SLIRS of stretch factor $s < 3/2$.

In [22] it is also mentioned that the problem of determining the minimum K such that a given weighted graph supports a shortest path IRS with a total number of K intervals, for the entire graph, is NP-hard.

Open Question 4
- Does the characterization of graphs supporting shortest path 1-SLIRS (and its variants) NP-complete when restricted to planar graphs?
- What is the best polynomial time approximation algorithm for the problem of characterization of graphs supporting a shortest path 1-SLIRS (and its variants)? Note that Theorem 14 implies that no polynomial time approximation algorithm can exist with a ratio less than 2 on the compactness of a graph.
- Does the following problem remain NP-hard: Find the smallest k such that k-stretch$(G) \leqslant s$, for some $s \geqslant 3/2$? or k-dilation$(G) \leqslant \delta$, for some δ?

In Section 4.1 we will see that the problem becomes polynomial under some assumptions of non-uniform link costs (dynamic link-costs).

The general problem of shortest path k-IRS is difficult to solve, however shortest path k-IRS have been found for many large classes of graphs.

3. Specific class of graphs

In this section we survey the results known about the compactness of graphs, that is the smallest integer k that provides a shortest path k-IRS. We start with some general results. Recall that the compactness of G is denoted by IRS(G), LIRS(G) (respectively

SIRS(G) and SLIRS(G)) denotes the linear compactness (respectively strict and strict-linear compactness). See Section 2.5.

For convenience, for every class of graphs $\mathscr{G}$, we denote by IRS($\mathscr{G}$) the value defined by

$$\mathrm{IRS}(\mathscr{G}) = \max_{G \in \mathscr{G}} \mathrm{IRS}(G)$$

that is a worst-case complexity of the compactness. A summary of the compactness of several classes of graphs is presented in Section 3.9. In the following n always denotes the number of nodes.

3.1. Extremal compactness

Because Interval Routing is devoted to implement routing in general networks, it is natural to wonder what is the compactness of an n-node graph.

The compactness of an n-node graph cannot exceed $n/2$, because a necessary condition to have k intervals for a set I is to have $|I| \leqslant n-k$, and clearly the number of intervals of I is at most $|I|$. The next result proposes a little improvement. The three next results use randomization.

Theorem 15 (Gavoille and Peleg [40]). *Every n-node graph G, $n \geqslant 1$, satisfies*

$$\mathrm{IRS}(G) < \frac{n}{4} + \frac{1}{4}\sqrt{2n\ln(3n^2)}.$$

Theorem 15 is actually a consequence of a deeper result: for every family $\mathscr{F}$ of at most $\mathrm{e}^{n/2}/n$ subsets of $\{1,\ldots,n\}$ there exists a permutation π of $\{1,\ldots,n\}$ such that all the subsets of $\mathscr{F}$ are composed of at most $n/4 + \mathrm{O}(\sqrt{n\ln n})$ intervals under π. It can be clearly applied on IRS models where Condition 2a and/or 2b (disjunction of the arc-labels) are relaxed, $\mathscr{F}$ representing the set of arcs, each subsets representing an arc-label, and the permutation π representing the node-labeling.

Due to the next lower bound, $n/4$ is asymptotically a tight bound.

Theorem 16 (Gavoille [37]). *For every n large enough, there exists an n-node graph G such that* IRS(G) $> n/4 - 1.72\, n^{2/3} \ln^{1/3} n$. *Moreover G is of diameter* 2, *and for every $k <$* IRS(G), k-dilation(G) $\geqslant 3$. *Therefore* k-stretch(G) $\geqslant 3/2$.

To a certain extent Interval Routing cannot be efficiently used for the *regional routing problem* posed by Peleg in [70], that consists in optimal routing up to distance k (cf. Definition 7 in Section 2.5).

Note that there exists a constructive proof for a worst-case compactness of at least $n/12$ [38]. A third proof, using randomization, for a $\Theta(n)$-lower bound is given in [53]. They also show that $\Theta(n)$ intervals might be required simultaneously on $\Theta(n)$ edges.

Against intuition, the next result shows that, in general, the compactness does not depend on the number of edges or on the maximum degree of the graph. It may be quite high even for sparse graphs.

Theorem 17 (Gavoille and Pérennès [42]). *For every n large enough, there exists a cubic graph G with at most n nodes such that* $\mathrm{IRS}(G) = \Theta(n)$. *Moreover* $\Theta(n)$ *intervals can be required on* $\Theta(n)$ *edges.*

As a corollary, there exists some graph with $(1+\varepsilon)n$ edges, for every $0 < \varepsilon \leqslant 1/2$, and of compactness $\Theta(\varepsilon n)$. Indeed, it suffices to choose a graph satisfying Theorem 17 with $2\varepsilon n$ nodes, and to connect $(1-2\varepsilon)n$ nodes of degree 1 to some nodes. Clearly such a new graph has n nodes, $3\varepsilon n + (1-2\varepsilon)n = (1+\varepsilon)n$ edges, and compactness $\Theta(\varepsilon n)$.

A way to prove Theorem 17 is to use the result of [43] about the *memory requirement* of a graph, which defines the smallest amount of memory needed to code any shortest path routing function (see [29] for an introduction to Universal Routing Schemes). In [43] n-node graphs of degree bounded by d, $3 \leqslant d \leqslant \varepsilon n$ for $\varepsilon < 1$, of local memory requirement $\Theta(n \log d)$ are constructed. This result, combined with Theorem 1, implies Theorem 17 (for $d = 3$ with a slightly modification to have a 3-regular graph).

We have also the following isolated results to quickly check whether a graph has compactness 1:

Theorem 18 (Gavoille and Guévremont [38])
- *Let G be an n-node graph with d nodes of degree $n-1$. Let m be the number of connected components of at least two nodes in the complement of G. If $d \geqslant (n-m)/2$ then* $\mathrm{SLIRS}(G) = 1$. *Therefore, every graph G obtained from K_n by removing c edges, for $c \leqslant (n+1)/4$, satisfies* $\mathrm{SLIRS}(G) = 1$.
- *Every graph G on $n < 7$ nodes, or on $m < 8$ edges satisfies* $\mathrm{SIRS}(G) = 1$. *Moreover there exists a counterexample of* 7 *nodes and* 8 *edges with compactness* 2 (*a cycle on* 6 *nodes with a path of length* 2 *connecting two nodes of the cycle at distance* 3).

3.2. Interconnection networks

We list in this paragraph the results about the graphs used for multi-processor based architectures.

Theorem 19 (Santoro and Khatib [68])
- SIRS(trees) = 1.
- SIRS(rings) = 1.

Theorem 20 (van Leeuwen and Tan [78])
- SLIRS(paths) = 1.
- SLIRS(2D-grids) = 1.
- SIRS(2D-grids with column-wrap-around) = 1.
- SLIRS(complete graphs) = 1.
- SIRS(complete bipartite graphs) = 1.

A more accurate form of the last point is presented in Theorems 39, and 40 in Section 3.7.

Theorem 21 (Bakker et al. [1])
- SLIRS(d-dimensional grids) = 1.
- SLIRS(hypercubes) = 1.
- SLIRS(d-dimensional tori) = 1, *if each dimension is of length at most* 4.

Applying the results of the product of graphs [30, 54] it is quite easy to show:

Theorem 22 (Kranakis et al. [54] and Fraigniaud and Gavoille [30])
- SLIRS(generalized hypercubes) = 1.
- SIRS(d-dimensional tori) = 1, *if the second largest dimension is of length at most* 4, *otherwise* SLIRS(d-dimensional tori) = 2.

Theorem 23 (Královič et al. [51])
- IRS(Shuffle-Exchange) = $\Omega(n^{1/2-\varepsilon})$, *for every* $\varepsilon > 0$.
- IRS(Cube Connected Cycles) = $\Omega(\sqrt{n/\log n})$.
- IRS(Butterfly) = $\Omega(\sqrt{n/\log n})$.
- IRS(Star graph) = $\Omega(n(\log\log n/\log n)^5)$.

Open Question 5
- Are the bounds of Theorem 23 tight?

There are some experimental results in [66] about the stretch factor and the average stretch factor of 2D-grids and Tori with different general labeling algorithms.

Theorem 24 (Sakho et al. [66]). *For every* $d \geqslant 4$, 1-stretch(d-dimensional tori) $\leqslant (d-2)/2$.

3.3. Chordal rings

A chordal ring is an augmented ring, or a *circulant graph* with a chord of length 1. Formally it is defined by the pair (n, L) where n is the number of nodes of the ring, and L is the set of chords, $L \subseteq \{2, \ldots, \lfloor n/2 \rfloor\}$. Each chord $l \in L$ connects every pair of nodes of the ring that are at distance l in the ring. We denote by $\mathscr{C}(n; \ell, \ldots, \ell_t)$, $l_1 \leqslant \cdots \leqslant l_t$, the chordal ring defined by $(n, \{l_1, \ldots, l_t\})$. The dimension of $\mathscr{C}(n; l_1, \ldots, l_t)$ is $t+1$. Let us emphasize that the degree of chordal rings is $2t$ in general, except whenever there is a chord of length $n/2$. In this case n is even and the degree is $2t-1$.

Theorem 25 (Narayanan and Opatrny [61])
- *For every* $l < n/2$, SIRS($\mathscr{C}(n; l)$) $\leqslant 2\sqrt{n}$.
- *For every* $l \leqslant \lceil \sqrt{2n-1} \rceil$, 1-dilation($\mathscr{C}(n; l)$) $\leqslant \lceil 3D/2 \rceil$, *where* D *is the diameter of* $\mathscr{C}(n; l)$.

A “natural” labeling of the nodes for chordal rings is the cyclic labeling: Consecutive labeling of nodes around the ring. This labeling is not always the best possible one. For instance, [55] showed that $\mathscr{C}(29;3)$ has no shortest path 1-IRS with cyclic labeling whereas it is of compactness 1. In fact it is possible to characterize graphs supporting a shortest path 1-IRS according a cyclic labeling of nodes.

Theorem 26
- (*Krizanc and Luccio* [55]) *A chordal ring* $\mathscr{C}(n;l)$ *has a shortest path* 1-*IRS with cyclic labeling if and only if* $n \bmod l = 0$, $n \bmod n-l = 0$, $n = sl+1$, *or* $n = sl-1$ *and* s *is odd*,[3] *or* $n = sl+2$ *and* s *is even.*
- (*Mans* [58]) *A chordal ring* $\mathscr{C}(n;l)$ *with* $n = sl-1$, s *is even and* l *odd, has a shortest path* 1-*SIRS with non-cyclic labeling.*

By the use of the Ádám property [58] showed that $\mathscr{C}(n;l)$ is isomorphic to $\mathscr{C}(n;l')$ if and only if $ll' \bmod n = 1$. For instance $\mathscr{C}(29,3)$, which has no shortest path 1-IRS with cyclic labeling (cf. [55]), is isomorphic to $\mathscr{C}(29;10)$ which satisfies Theorem 26. Therefore, $\mathrm{SIRS}(\mathscr{C}(29;3)) = 1$.

Conjecture 1 (Mans [58]). *If* $\mathrm{SIRS}(\mathscr{C}(n;l)) = 1$ *then either* $\mathscr{C}(n;l)$, *or* $\mathscr{C}(n;l')$ *has a shortest path* 1-*SIRS with cyclic labeling*, *with* $ll' \bmod n = 1$.

Theorem 27
- (*Mans* [58]) *For every* $l \leqslant 3$, $\mathrm{SIRS}(\mathscr{C}(n;l)) = 1$.
- (*Mans* [58]) *For every* $i \geqslant 2$, *and every* $j \geqslant 1$, $\mathrm{SIRS}(\mathscr{C}(i(4j+1);2i)) > 1$.
- (*Narayanan and Opatrny* [61]) *For every* $i \geqslant 2$, $\mathrm{SIRS}(\mathscr{C}(2i^2+2i+1;2i+1)) > 1$.

Let $\mathscr{C}_i = \mathscr{C}(2i^2+2i+1;2i+1)$. Since $\mathscr{C}_i$ is of diameter i (cf. [3]), it follows that 1-stretch$(\mathscr{C}_i) \geqslant 1+1/i$, for every $i \geqslant 2$. (Actually, $\mathscr{C}_i$ is the chordal ring of degree 4 with the largest number of nodes for a given diameter). Moreover, in [61], it is shown that 1-stretch$(\mathscr{C}_i) \leqslant 2$. Also, the smallest chordal ring known to be of compactness greater than 1 is $\mathscr{C}(13,5)$, and 1-stretch$(\mathscr{C}(13,5)) \geqslant 3/2$. [61] showed that there exists some chordal rings, $\mathscr{C}_i$, where every shortest path IRS using a cyclic labeling requires $\sqrt{n/2}$ intervals, and also that $\mathscr{C}(35;5)$ has no shortest path 1-IRS of stretch factor $\leqslant 2$ with a cyclic labeling. Other results about directed chordal rings are mentioned in [55].

Open Question 6
- Is there some chordal ring of compactness $\Omega(\sqrt{n})$?
- Characterize chordal rings of compactness k with cyclic labeling, for $k \geqslant 2$.

For two chords we have:

[3] The Theorem was unfortunately mistyped in the original paper. Here, we give the correct statement.

Theorem 28 (Flammini et al. [21]). *Let i, j be two integers, and $C = \mathscr{C}(n; i, j)$.*
- *if* $n \bmod i = 0$, *and* $n \bmod j = 0$, *then* $\mathrm{IRS}(C) = 1$.
- *if* $j \bmod i = 0$, $n \bmod j = r \neq 0$, *and* $n/j \geqslant \max\{j - r - 1, r\}$, *then* $\mathrm{IRS}(C) = \mathrm{O}(\min\{n, j^2\}/i)$.
- *if* $n \bmod j = 0$, *and* $j = i + 1$, *then* $1\text{-stretch}(C) \leqslant 3/2$.
- *if* $j = i + 1$, *or* $2j < i$, $1\text{-stretch}(C) \leqslant (i + 2)/4$.
- *if* $2i > j$, *then* $1\text{-stretch}(C) \leqslant (j - i + 2)/2$.

And for higher dimensions,

Theorem 29 (Flammini et al. [21])
- *For every* $i \geqslant 1$, $\mathrm{IRS}(\mathscr{C}(n; 1, \ldots, i)) = 1$.
- *For every* $r, p \geqslant 2$, *and for every subset* $\{l_1, \ldots, l_t\} \subseteq \{1, \ldots, p - 1\}$, $\mathrm{IRS}(\mathscr{C}(r^p; r^{l_1}, \ldots, r^{l_t})) = 1$.

The latter result can be easily generalized to $\mathscr{C}(n; l_1, \ldots, l_t)$ when $l_i \bmod n = 0$, and $l_{i+1} \bmod l_i = 0$ for every i.

In [21] is presented several results concerning IRS without the disjointness property of the labels assigned to the arcs, i.e., with Condition 2b of Definition 1 relaxed. See Section 4.3 for a discussion of these results.

Note also that most of the results extend to circulant graphs in general, i.e., does not need to specify the chord of length 1.

Open Question 7
- Is there a general upper bound for the compactness of chordal rings of dimensions d of the form $n^{1-\mathrm{O}(1/d)}$?

3.4. Planar graphs

Let us recall that a *plane* graph is a planar graph embedded in the plane without any crossing edges. An *outerplanar* graph is a plane graph with all its nodes lying on one face (this includes trees).

Theorem 30 (Frederickson and Janardan [32]). *Every plane graph G satisfies* $\mathrm{SIRS}(G) \leqslant 3p/2$, *where p is the smallest number of disjoint faces that cover all the nodes. Therefore, every outerplanar graph G satisfies* $\mathrm{SIRS}(G) = 1$.

Unfortunately the number of disjoint faces, p, can attain $\Theta(n)$. In [32] the result is generalized for graph of genus γ with an upper bound of $3p/2 + \gamma$. Other results, but slightly weaker, are presented in the same article. No bound has been proved yet to be tight.

Theorem 31 (Gavoille and Pérennès [42]). *For every integer n large enough there exists*:

- *an n-node planar graph of compactness at least* $\sqrt{n}/2.6$;
- *a cubic planar graph of at most n nodes of compactness at least* $\sqrt{n}/5.6$;
- *an n-node planar graph on which every shortest path IRS has a total number of intervals required for the entire graph at least* $\Omega(n^{3/2})$, *i.e.*, $\Omega(\sqrt{n})$ *intervals on a constant fraction of all the edges*;
- *an n-node triangulated*[4] *plane graph of bounded degree of compactness* $\Omega(\sqrt{n})$.

The following conjecture already appears as open question in [36].

Conjecture 2. *Every n-node planar graph has compactness* $O(\sqrt{n})$.

Theorem 32 (Tse and Lau [75]). *For every even integer* $D \geqslant 2$ *there exists an n-node planar graph G of diameter D such that*
- k-dilation$(G) \geqslant (1+1/(2k))D-1$, *for every* $k \leqslant (n/24)^{1/3}$;
- k-dilation$(G) \geqslant (1+1/(4k))D-1$, *for every* $k \leqslant (n/38)^{1/2}$.

The latter result uses a *globe-graph* (a subdivision of a complete bipartite graph $K_{2,t}$) which is a series-parallel graph [11], and also a graph of treewidth two [4]. Thus the general $\Omega(\sqrt{n})$-lower bound for planar graphs holds also for the subclass of graphs of treewidth two (see Section 3.5 for results about treewidth).

The best known lower bound of compactness, regarding the multiplicative constant, for series-parallel graphs is:

Theorem 33 (Královič et al. [52])
- *For every integer* $t \geqslant 1$, *there is a series-parallel graph G of* $n = t(t+1)+2$ *nodes such that* $\mathrm{IRS}(G) \geqslant t/4 > \sqrt{n}/4 - 1/2$ (*a globe-graph with t paths of length* $t+2$).
- *There are bounded degree series-parallel graphs of compactness* $\Omega(\sqrt{n})$ (*a* $\sqrt{n} \times \sqrt{n}$ *grid where all internal and horizontal edges have been removed*).

Historically, the first $\Omega(\sqrt{n})$-lower bound for a series-parallel graph has been implicitly proved by [72]. The previous bound (using the same graph) was of $\Omega(n^{1/3})$ in [54].

Open Question 8
- Are there constants k and c such that every planar graph G of diameter D satisfies k-dilation$(G) \leqslant 3D/2 + c$? or k-stretch$(G) \leqslant c$?

Actually [52] proposed a stronger version of Question 8: For every $\varepsilon > 0$, is there a constant k such that every planar graph G of diameter D satisfies k-dilation$(G) \leqslant (1+\varepsilon)D$?

[4] A plane graph where all the faces are triangles.

3.5. Treewidth bounded graphs

The next result confirms the lower bound of Theorem 33, and gives a generalization. Note that graphs of treewidth 1 are trees, and therefore have compactness 1 by Theorem 3.

Theorem 34. *For every k, $2 \leqslant k \leqslant (n/3)^{1/3}$, there exists an n-node graph G of treewidth k such that* $\mathrm{IRS}(G) = \Omega(\sqrt{kn})$ *(a subdivision of the graph $K_{k,r}$ with $r = \Theta(\sqrt{kn})$).*

Proof. Let k, r, s be three integers such that $r \geqslant k \geqslant 2$, and $l \geqslant 1$. Consider the graph G obtained by subdividing all the edges in $2l+1$ nodes of the bipartite graph $K_{k,r}$. G has $n = k + r + (2l+1)kr$ nodes and a treewidth $\min\{k,r\} = k$, see [49, Lemma 2.2.1, p. 18]. We consider a node x initially belonging to the k-partition of $K_{k,r}$, and a node y initially belonging to the r-partition of $K_{k,r}$. Let W be the set of all the nodes at an odd distance of x. A set A of arcs is composed of all the incident arcs from x, all the incident arcs of y, and all the arcs of the directed path going from x to y and whose tail is at an even distance from x.

The nodes of W and the arcs of A constitute a *matrix of constraints* of G, i.e., the shortest path from the tail of any arc $(a,b) \in A$ to any node of $w \in W$ is unique (see [23, 38] for a formal description of this technique): we put "1" in such a boolean matrix if the route from a to w must use the arc (a,b); "0" otherwise. The dimension of the matrix is $|W| \times |A|$. We consider the submatrix M obtained by removing the rows corresponding to nodes of W belonging to the subdivision of the induced subgraph $K_{1,r}$ rooted in y. Therefore M has $|W| - (s+1)r = (k-1)r(s-1)$ rows and $|A| = k + r + s$ columns.

Every two rows of M differ by at least one place (note that this is false if we take all of W as the set of nodes). Therefore there exists some arc $(a,b) \in A$ such that the set $\mathscr{I}(a,b)$ is composed of at least $I = (|W| - (s+1)r)/(2|A|)$ intervals, whatever the shortest path IRS $(\mathscr{L}, \mathscr{I})$. Indeed, the total number of intervals for all the arcs of A is at least $(|W| - (s+1)r)/2$, the number of 01-sequences for the columns of the matrix under row permutation (each such sequence begins a new interval).

Let $r = s$. This is possible because in this case $n = 2kr^2 + (k+1)r + k$; so it suffices to choose any integer $r \geqslant \sqrt{n/(3k)}$ in order to have a graph with at most n nodes, and also $r \geqslant k$ for the desired range for k: $2 \leqslant k \leqslant (n/3)^{1/3}$.

$$I = \frac{|W| - (s+1)r}{2|A|} = \frac{(k-1)r(s+1)}{2(k+r+s)} > \frac{(k-1)r^2}{2(k+2r)}.$$

Since $r \geqslant k$, $k + 2r \leqslant 3r$, and thus $I > (k-1)r/6$. On the other hand $n = 2kr^2 + r(k+1) + k < 8(k-1)r^2$, for every $r \geqslant k \geqslant 2$. Thus $r > \sqrt{n/(8(k-1))}$. We get finally that

$$I > \frac{k-1}{6}\sqrt{\frac{n}{8(k-1)}} = \frac{1}{6\sqrt{8}}\sqrt{(k-1)n},$$

which is $\Omega(\sqrt{kn})$ for every $k \geqslant 2$. $\square$

Theorem 35 (Narayanan and Nishimura [60])
- SIRS(2-trees) = 3 (thus there exists some 2-trees of compactness 3).
- SIRS(k-trees) $\leqslant 2^{k+1}$.

Clearly the first result does not hold for partial 2-trees (a 2-tree obtained by deleting some edges), because it is well-known that partial 2-trees have treewidth at most 2, and compactness $\Omega(\sqrt{n})$ by Theorem 34. The second result is particularly interesting since, in general, k is a constant in comparison with n, the number of nodes.

In [25], there is another study covering 1-IRS on directed series-parallel graphs. They construct a shortest path 1-IRS for the directed case, and gave also a valid 1-ILS for the undirected case, however without explicit bound on the routing path length for the undirected case.

Open Question 9
- Is the bound SIRS(k-trees) $\leqslant 2^{k+1}$ tight?
- Is there some function f such that IRS($\mathscr{T}_k$) $\leqslant f(k)\sqrt{n}$, where $\mathscr{T}_k$ is the class of graphs of treewidth at most k?

Note that even for the class of graphs having O(n) there exists some graphs with $1.18n$ edges (actually almost every n-node graph with $m \geqslant 1.18n$ edges) and treewidth $\Theta(n)$ [49, Theorem 5.3.2, p. 58], and also bounded degree graphs of compactness $\Theta(n)$ (Theorem 17).

Conjecture 3. *Every n-node graph of constant treewidth has compactness* o(n).

Several works have been done about the trade-off between the dilation and the compactness for the class of *multi-globe* graphs. A multi-globe graphs is a subdivision of the graph $K_{a,b}$ in such a way that all the edges of $K_{a,b}$ are subdivided in the same number of nodes. This class provides a large set of counterexamples (actually used in Theorems 7, and 34) and includes the globe-graphs (used in Theorems 32, and 33).

For convenience, for every class of graphs $\mathscr{G}$, we will denote by k-dilation($\mathscr{G}$) = $\max_{G \in \mathscr{G}}$ k-dilation(G), and similarly for k-stretch($\mathscr{G}$).

Theorem 36
- (*Tse and Lau* [74]) 1-dilation(multi-globe) $\geqslant 2D - 3$.
- (*Královič et al.* [52])[5] 3-dilation(multi-globe) $\leqslant 1.25D$.
- (*Královič et al.* [52]) k-dilation(multi-globe) $\leqslant 1.25D - 1$ *implies* $k = \Omega(\sqrt{n})$.
- (*Královič et al.* [52]) $\lceil \sqrt{n}/(2\varepsilon) \rceil$-dilation(multi-globe) $\leqslant (1+\varepsilon)D$, *for every* $\varepsilon > 0$.
- (*Královič et al.* [52]) 1-dilation(globe) $\leqslant 1.5D$.
- (*Ružička* [64]) k-dilation(globe) $\leqslant 1.5D - 1$ *implies* $k > 1$.

[5] The original result presented in this article is 2-dilation(multi-globe) $\leqslant 1.25D$. However in the construction given in the proof it appears that the number of intervals is 3. To our best knowledge we do not know whether it can be done with 2 intervals only.

- (*Tse and Lau* [72]) $O(1/\varepsilon)$-dilation(globe) $\leqslant (1+\varepsilon)D$, *for every* $\varepsilon > 0$.
- (*Fraigniaud et al.* [31]) 1-stretch(globe) $\in [3 - O(1/D), 3]$.

Note that, in most of the articles cited in Theorem 36, the diameter D of globe or multi-globe graphs is even.

3.6. Random graphs

The class $\mathscr{G}_{n,p}$ denotes the classic model of n-node random graphs, where p represents the probability to have an edge between any two nodes.

Theorem 37 (Flammini et al. [24])
- *Let* $p = n^{-1+1/s}$ *for an integer* $s > 0$ *such that there exists* ε *that satisfies* $(\ln^{1+\varepsilon} n)/n < p < n^{-1/2-\varepsilon}$. *Then, given a graph* $G \in \mathscr{G}_{n,p}$, $\mathrm{IRS}(G) = \Omega(n^{1-6/\ln(np) - \ln(np)/\ln n})$ *with probability* $1 - o(1)$.
- *Given a graph* $G \in \mathscr{G}_{n,p}$, *for some* $p = n^{-1+1/\Theta(\sqrt{\log n})}$,

$$\mathrm{IRS}(G) = \Omega(n^{1-1/\Theta(\sqrt{\log n})}) \quad \text{with probability } 1 - o(1).$$

The next result includes the case $p = 1/2$, and shows that almost all n-node graphs have constant compactness.

Theorem 38 (Gavoille and Peleg [41]). *For sufficiently large* n, *with probability at least* $1 - 1/n^2$, *and for every fixed* p, $0.45 < p < 1$, *a random graph* $G \in \mathscr{G}_{n,p}$ *satisfies* $\mathrm{SLIRS}(G) = 2$. *Moreover, with probability at least* $1 - 1/n$, G *has a shortest path* 2-*SLIRS using a single interval per arc, except for* $O(\log^3 n)$ *arcs per node.*

Open Question 10
- Do almost all n-node graphs have compactness 1?

3.7. Other classes of graphs

$K_{n_1,\ldots,n_r}$ denotes the complete r-partite graph of sets of nodes $V = V_1 \cup \cdots \cup V_r$, $n_i = |V_i|$. The edges are such that for every $i \neq j$, the graph induced by V_i and V_j is a complete bipartite graph, K_{n_i,n_j}. For instance, $K_{1,\ldots,1} = K_n$.

Theorem 39 (Kranakis et al. [54]). *For every* $2 \leqslant n_1 \leqslant \cdots \leqslant n_r$, $\mathrm{LIRS}(K_{n_1,\ldots,n_r}) = 1$.

This has been slightly improved later for bipartite graphs (improving also a result of Theorem 20):

Theorem 40 (Narayanan and Shende [62])
- *For every* $1 \leqslant n_1 \leqslant n_2$, $\mathrm{SIRS}(K_{n_1,n_2}) = 1$.
- *For every* $2 \leqslant n_1 \leqslant n_2$, $\mathrm{SLIRS}(K_{n_1,n_2}) = 1$.

The next class of graphs includes complete graphs, paths and rings. See [44] for an introduction to these classes of graphs. Note that the class of unit interval graphs corresponds with the class of proper interval graphs.

Theorem 41 (Fraigniaud and Gavoille [30])
- SIRS(unit arc-circular graphs) $= 1$, *and* IRS(arc-circular graphs) > 1.
- SLIRS(unit interval graphs) $= 1$, *and* LIRS(interval graphs) > 1.

The latter result has been extended to the interval graphs (this class includes trees):

Theorem 42 (Narayanan and Shende [62]). SIRS(interval graphs) $= 1$.

By an exhaustive computation of all the possible labeling, we found that the Petersen graph, P, satisfies $\mathrm{SLIRS}(P) = \mathrm{LIRS}(P) = \mathrm{SIRS}(P) = \mathrm{IRS}(P) = 3$. $P = (V, E)$ is an intersection graph [6] defined by $V = \{X \subseteq \{1, \ldots, 5\} \mid |X| = 2\}$.

Finally, there are several works on Interval Routing with experimental approaches and simulation results: [6, 48, 47, 57, 69].

3.8. Graph operators

There are several results about some graph operators: Cartesian product, composition, and join of graphs.

The *Cartesian product* of $G_1 = (V_1, E_1)$ with $G_2 = (V_2, E_2)$, denoted by $G_1 \times G_2$, has the node set $V_1 \times V_2$, and the edge set $\{(u, v) \mid u = (u_1, u_2), v = (v_1, v_2)$, and $[(u_1 = v_1$ and $(u_2, v_2) \in E_2)$ or $(u_2 = v_2$ and $(u_1, v_1) \in E_1)]\}$.

The first point (the linear case) of Theorem 43 is due to [54], the others due to [30]. The lower bounds of this theorem are obtained by application of Theorem 52 in Section 4.1.

Theorem 43 (Kranakis et al. [54]; Fraigniaud and Gavoille [30])
- $\mathrm{SLIRS}(G \times H) = \max\{\mathrm{SLIRS}(G), \mathrm{SLIRS}(H)\}$;
- $\max\{\mathrm{LIRS}(G), \mathrm{LIRS}(H)\} \leqslant \mathrm{LIRS}(G \times H) \leqslant \max\{\mathrm{LIRS}(G), \mathrm{SIRS}(H)\}$;
- $\max\{\mathrm{SIRS}(G), \mathrm{SIRS}(H)\} \leqslant \mathrm{SIRS}(G \times H) \leqslant \max\{\mathrm{SLIRS}(G), \mathrm{SIRS}(H)\}$;
- $\max\{\mathrm{IRS}(G), \mathrm{IRS}(H)\} \leqslant \mathrm{IRS}(G \times H) \leqslant \max\{\mathrm{LIRS}(G), \mathrm{SIRS}(H)\}$.

We can see one motivation to distinguish the strictness and the linearity of IRS. To apply Theorem 43, one graph must support a strict IRS and the other a linear IRS. Note that, in general, $\mathrm{IRS}(G \times H) \neq \max\{\mathrm{IRS}(G), \mathrm{IRS}(H)\}$. The 5×5-torus is a counterexample. In [30], other results are mentioned about the k-dilation in Cartesian product.

[6] I.e., $E = \{(A, B) \in V^2 \mid A \neq B$ and $A \cap B \neq \emptyset\}$.

Open Question 11
- Does $\mathrm{LIRS}(G) \neq \mathrm{IRS}(G)$ imply $\mathrm{IRS}(G \times C_n) \neq \mathrm{IRS}(G)$ for every $n \geqslant 5$? (where C_n denotes an n-node cycle)

The *composition* of $G_1 = (V_1, E_1)$ with $G_2 = (V_2, E_2)$, denoted by $G_1[G_2]$, has the node set $V_1 \times V_2$, and the edge set $\{(u,v) \mid u = (u_1, u_2), v = (v_1, v_2)$, and either $(u_1, v_1) \in E_1$ or $(u_1 = v_1$ and $(u_2, v_2) \in E_2)\}$. The previous result of [54] has been improved with:

Theorem 44 (Narayanan and Shende [62])
- $\mathrm{SLIRS}(G[H]) \leqslant \mathrm{SLIRS}(G) + 1$.
- *If H is a unit interval graph, then* $\mathrm{SIRS}(G[H]) \leqslant \mathrm{SIRS}(G)$.

In the following, $\mathrm{SLIRS}^{\mathrm{tree}}(G)$ denotes the smallest k such that G supports an k-SLIRS such that from each node the routing paths induced by the k-SLIRS form a tree (see after Open Question 14, in Section 4.6).

Theorem 45 (De la Torre et al. [10]). *For any H,* $\mathrm{SLIRS}^{\mathrm{tree}}(G[H]) \leqslant \mathrm{SLIRS}^{\mathrm{tree}}(G)$.

The *join* of $G_1 = (V_1, E_1)$ with $G_2 = (V_2, E_2)$, denoted by $G_1 + G_2$, has the node set $V_1 \cup V_2$, and the edge set $E_1 \cup E_2 \cup \{(u,v) \mid u \in V_1, v \in V_2\}$.

Theorem 46 (Kranakis et al. [54]). *Suppose G_1 and G_2 are graphs with n_1 and n_2 nodes, minimum degrees δ_1 and δ_2, and maximum degrees Δ_1 and Δ_2 respectively. Then,*

$$\mathrm{LIRS}(G_1 + G_2) \leqslant 1 + \max\{\lceil f_1/n_2 \rceil, \lceil f_2/n_1 \rceil\}$$

where $f_i = \min\{n_i - \delta_i - 1, \Delta_i + 1\}, i \in \{1, 2\}$.

3.9. Summary

In Table 1, only shortest paths IRS are considered, and n denotes the number of nodes of the graph.

4. Variations on the interval routing model

4.1. Fixed and dynamic link-cost models

In [32] the first results dealing with weighted graphs and dynamic link-cost model are presented. Their nice characterization for compactness 1 has been extended in [1] for the linear case, and finally [5] proved that a linear time algorithm exists for the characterization of graphs of compactness k, for every fixed $k \geqslant 1$, in the setting of dynamic link-cost model.

Table 1

IRS classes	Graphs
1-SLIRS	Paths, complete graphs, d-dimensional grids, d-dimensional tori with each dimension of length $\leqslant 4$, complete bipartite graphs $K_{p,q}$ with $p,q \geqslant 2$, unit interval graphs, hypercube, generalized hypercube
1-SIRS	Trees, rings, outerplanars, interval graphs, d-dim. tori with the 2nd largest dim. of length $\leqslant 4$, 2D-grids with column-wrap around, graphs with $n \leqslant 6$, chordal ring of degree 4 with chord length $\leqslant 3$, unit arc-circular graphs, $K_{p,q}$ with $p,q \geqslant 1$
2-SLIRS	d-dimensional tori, almost all graphs
3-SIRS	2-trees, Petersen graph
2^{k+1}-SIRS	k-trees
$\lfloor 3p/2 + \gamma \rfloor$-SIRS	p-plane graphs of genus γ
$\Omega(n^{1/2-\varepsilon})$-IRS	Shuffle-exchange, Cube connected cycles, Butterfly
$\lfloor 2\sqrt{n} \rfloor$-SIRS	Chordal rings of degree 4
$\Omega(\sqrt{n})$-IRS	Cubic planar, triangulated planar, series-parallel graphs
$\Omega(\sqrt{kn})$-IRS	Graphs of treewidth k, $2 \leqslant k \leqslant (n/3)^{1/3}$
$\Omega(f(n))$-IRS	Star graph, with $f(n) = n((\log\log n)/\log n)^5$
$\Omega(m-n)$-IRS	Worst-case graph with m edges, $n \leqslant m \leqslant 3n/2$
$\Omega(n)$-IRS	Worst-case cubic graph
$\alpha(n)$-SLIRS	All graphs, with $\alpha(n) < n/4 + (1/4)\sqrt{2n\ln(3n^2)}$

In this paragraph we consider weighted graphs, that is a pair (G,w) where $G=(V,E)$ is a graph, and w a non-negative function that assigns a "weight" for each edge (the weight is the same for $(x,y) \in E$ and $(y,x) \in E$). This weight can take into account the communication costs along the link between x and its neighbor y. The *cost of a path* in a weighted graph is the sum of the weights of the arcs that compose the path. A *minimum path* is a path of minimum cost between its two end-points.

Definition 9. A graph G supports a *minimum fixed cost* IRS if for every weight function w of G there is an IRS on G such that all the routing paths are minimum paths. The class of graphs having a minimum fixed cost k-IRS is denoted by k-$\mathscr{IRS}_f$.

Let us denote by k-$\mathscr{IRS}$ the class of graphs which support a shortest path k-IRS. Clearly k-$\mathscr{IRS}_f \subseteq k$-$\mathscr{IRS}$ because the uniform weight function is a particular case of the model k-$\mathscr{IRS}_f$.

Another model assumes that the addresses of the nodes are fixed once in advance (independent of the link costs), and then the weights can change. This model takes into account the situation where the link-costs evolve over time, and where the computation of the node-labeling through the network is impossible, whereas the edge-labeling remains locally adjustable in order to achieve a routing path of minimal length.

Definition 10. A graph G supports a *minimum dynamic cost* IRS if there exists a node-labeling of G, $\mathscr{L}$, such that for every weight function w of G there is an IRS on G with respect to $\mathscr{L}$ for which all the routing paths are minimum paths. The class of graphs having a minimum dynamic cost k-IRS is denoted by k-$\mathscr{IRS}_d$.

Similarly, k-$\mathscr{IRS}_d \subseteq k$-$\mathscr{IRS}_f$ because for $G \in k$-$\mathscr{IRS}_d$, the node-labeling $\mathscr{L}$ gives a labeling solution for every weight function w. Hence k-$\mathscr{IRS}_d \subseteq k$-$\mathscr{IRS}_f \subseteq k$-$\mathscr{IRS}$, and all the lower bounds proved in the uniform link-cost model hold also for the two other models. We extend the previous notation to the linear and/to strict IRS in a similar way.

The dynamic link-cost model assumes implicitly that addresses of the nodes cannot be permuted in order to optimize the number of intervals per edge with the routing path lengths. It turns out that Theorem 15 (compactness at most $n/4 + o(n)$ for every graph) holds for the fixed link-cost model, but not for the dynamic link-cost model.

For convenience, a statement which holds independently for all the classes k-$\mathscr{IRS}$, k-$\mathscr{IRS}_f$, and k-$\mathscr{IRS}_d$ is termed with k-$\mathscr{IRS}_\star$. By definition, we set $K_1 \in 0$-$\mathscr{SLIRS}_\star$.

Theorem 47.
- *For every n-node graph G, $n \geqslant 1$,*
  - $G \in \lceil n/2 \rceil$-$\mathscr{S}LIRS_\star$;
  - $G \in \lfloor n/2 \rfloor$-$\mathscr{S}IRS_\star$;
  - $G \in \lfloor n/2 \rfloor$-$\mathscr{L}IRS_\star$;
  - $G \in (\lceil n/2 \rceil - 1)$-$\mathscr{IRS}_\star$.
- *For every n, $n \geqslant 1$,*
  - $K_n \in k$-$\mathscr{S}LIRS_d$ *implies* $k \geqslant \lceil n/2 \rceil$;
  - $K_n \in k$-$\mathscr{S}IRS_d$ *implies* $k \geqslant \lfloor n/2 \rfloor$;
  - $K_n \in k$-$\mathscr{L}IRS_d$ *implies* $k \geqslant \lfloor n/2 \rfloor$;
  - $K_n \in k$-$\mathscr{IRS}_d$ *implies* $k \geqslant \lceil n/2 \rceil - 1$.

Proof. Let us start to prove the first point. Let $R = (\mathscr{L}, \mathscr{I})$ be an IRS on G. Consider $I = \mathscr{I}(x, y) \subseteq \{1, \ldots, n\}$. Let k be the number of intervals of I. Let $I' = \{1, \ldots, n\} - I$ be the complement of I, and k' its number of intervals. Clearly, $k + k' \leqslant n$. Moreover, for the linear cases $k \leqslant k' + 1$, and $k = k'$ for cyclic intervals. It implies that

$$2k \leqslant k + k' + 1 \leqslant n + 1 \;\; \Rightarrow \;\; k \leqslant \left\lfloor \frac{n+1}{2} \right\rfloor = \left\lceil \frac{n}{2} \right\rceil,$$

for the linear cases, and $2k = k + k' \leqslant n$ which implies $k \leqslant \lfloor n/2 \rfloor$ for the cyclic cases. It remains to show the non-strict cases.

Assume R is non-strict and k is minimum. To show that $G \in \lfloor n/2 \rfloor$-$\mathscr{LIRS}_\star$, it suffices to show that for n odd the previous bound decreases by 1. Assume $n = 2i + 1$, and that I is composed of $\lceil n/2 \rceil = i + 1$ intervals. Note that I' is necessarily composed of at least i intervals. Moreover all the intervals that composed I and I' are single, i.e., of the form $[a]$. Indeed if 1 interval (at least) is composed of 2 integers (or more) then the number of elements in $I \cup I'$ would be $(i+1)+i+1=2i+2=n+1$ which is impossible. If $\mathscr{L}(x) \in I$ then $\mathscr{L}(x)$ appears as a single interval, so removing it will save one interval. I contains 1 and n, because I and I' are composed of single intervals only, and $|I| > |I'|$. If $\mathscr{L}(x) \notin I$ then its insertion will win one interval because $\mathscr{L}(x)-1$ and $\mathscr{L}(x)+1$ are both single intervals and both belong to I (we can assume $n \geqslant 3$ because for $n \leqslant 2$ the result is trivial).

To prove $G \in (\lceil n/2 \rceil - 1)$-$\mathscr{IRS}_\star$ it suffices to prove it for n even. So assume $n=2i$, and I composed of $\lfloor n/2 \rfloor = i$ intervals. Similarly I and I' are composed of single intervals only, and thus the insertion or deleting of $\mathscr{L}(x)$ in I will decrease by 1 the number of intervals of I.

Let us show the second point. Consider the nodes labeled 1 and 2. First consider the case of strict intervals ($\mathscr{SLIRS}_d$ or $\mathscr{SIRS}_d$). Choose w such that $w(2,1)=1$, $w(1,2i+1)=1$ for i integer such that $3 \leqslant 2i+1 \leqslant n$, and $w(x,y)=3$ otherwise. The shortest paths from 2 to i travels to 1 if i is odd (cost $\leqslant 2$, and 3 otherwise). Thus $\mathscr{I}(2,1)=\{1,3,\ldots,2i+1,\ldots\}$. It is composed of $|\mathscr{I}(2,1)| = \lceil n/2 \rceil$ linear intervals (there are no consecutive integers in $\mathscr{I}(2,1)$), and $\lfloor n/2 \rfloor$ cyclic intervals (in the odd case 1 and n can be merged).

For the non-strict intervals ($\mathscr{LIRS}_d$ or $\mathscr{IRS}_d$), choose w such that $w(2,1)=1$, $w(1,2i)=1$ for i integer such that $4 \leqslant 2i \leqslant n$, and $w(x,y)=3$ otherwise. The shortest paths from 2 to i travels to 1 if i is even ($i \neq 2$). Hence $\mathscr{I}(2,1) \subseteq \{1,4,\ldots,2i,\ldots\}$. $\mathscr{L}(2)$ can belong to $\mathscr{I}(2,1)$ or not, it does not change the number of intervals of $\mathscr{I}(2,1)$. It is composed of $|\mathscr{I}(2,1)| = \lfloor n/2 \rfloor$ linear intervals, and $\lfloor (n-1)/2 \rfloor = \lceil n/2 \rceil - 1$ cyclic intervals. This completes the proof. □

The inherent definition of the link-cost model imposes that a graph can have very different compactness depending on the model (uniform, fixed or dynamic link-cost). Indeed we have seen in Theorem 20 that $K_n \in 1$-$\mathscr{SLIRS}$. In the opposite to the class k-$\mathscr{IRS}$ the fixed and dynamic link-cost models have known characterizations.

The definition of outerplanar graphs is recalled in Section 3.4.

Theorem 48 (Frederickson and Janardan [32]). *$G \in 1$-$\mathscr{SIRS}_d$ if and only if G is outerplanar.*

Theorem 49 (Bakker et al. [1]). *$G \in 1$-$\mathscr{IRS}_d$ if and only if every 2-vertex-connected component of G is outerplanar or K_4.*

A *G-star* is a graph obtained from G by adding zero or more nodes of degree one to the nodes of G. A *centipede* is a P-star, P being a path where some edges of P are replaced by a K_3 (the edge of the path is identified to an edge of K_3). A Y-graph is a tree on 7 nodes with one node of degree 3, and three nodes of degree 2. Finally, H is a *subgraph of minimum paths* of a weighted graph G if H is a subgraph of G, and if for all $x,y \in V(H)$, all the minimum paths between x and y in G are wholly contained in H. For unweighted graph G, H is called *subgraph of shortest paths*.

Theorem 50 (Bakker et al. [1])
- $G \in 1\text{-}\mathscr{L}\mathscr{I}\mathscr{R}\mathscr{S}_f$ *if and only if G is a centipede, a* K_3*-star or a* K_4*-star that does not contain a Y-graph as subgraph of minimum paths.*
- $G \in 1\text{-}\mathscr{L}\mathscr{I}\mathscr{R}\mathscr{S}_d$ *if and only if G is a centipede.*

According to [79] we have:

Theorem 51 (Bakker et al. [2]). $G \in 1\text{-}\mathscr{S}\mathscr{L}\mathscr{I}\mathscr{R}\mathscr{S}_d$ *if and only if G is a path.*

The following result proposes a tool to show that a given graph does have not compactness k. It suffices to show that one of its subgraphs of minimum paths is not.

Theorem 52 (Fraigniaud and Gavoille [30]). *A given graph belongs to* $k\text{-}\mathscr{I}\mathscr{R}\mathscr{S}_\star$ *if all its subgraphs of minimum paths belongs to* $k\text{-}\mathscr{I}\mathscr{R}\mathscr{S}_\star$ (*the same holds for the linear and/or strict variants*).

This is also a sufficient condition since G is a subgraph of minimum paths of itself. In the original paper, the result has been mentioned only for the uniform link-cost model. The proof holds for all the models, and can be easily adapted for a version using stretch factor and dilation.

An interesting application of Theorem 52, and the lithium-based characterization of 1-linear IRS (cf. Theorem 5) is that every graph that contains a weak-lithium graph, e.g., $K_{1,3}$, as subgraph of shortest paths does not support any shortest path 1-SLIRS.

Because IRS (with cyclic intervals) are invariant under shifting of the labels modulo n, it is easy to compose graph by union sharing one cut-vertex. The sufficient condition of the next result is an application of Theorem 52.

Theorem 53 (Gavoille and Guévremont [38]). *For every* $k \geqslant 1$, $G \in k\text{-}\mathscr{I}\mathscr{R}\mathscr{S}_\star$ (*respectively* $k\text{-}\mathscr{S}\mathscr{I}\mathscr{R}\mathscr{S}_\star$) *if and only if all its* 2*-vertex-connected components are in* $k\text{-}\mathscr{I}\mathscr{R}\mathscr{S}_\star$ (*respectively* $k\text{-}\mathscr{S}\mathscr{I}\mathscr{R}\mathscr{S}_\star$).

Here a version for the linear case, and for $k = 1$:

Theorem 54 (Narayanan and Shende [62]). *Let G be a graph, let* $G_1, \dots, G_t$ *be its* 2*-vertex-connected components, and let* $x_{i,j}$ *be the cut-vertex between the component* G_i *and* G_j, $1 \leqslant i < j \leqslant t$. $G \in 1\text{-}\mathscr{S}\mathscr{L}\mathscr{I}\mathscr{R}\mathscr{S}_\star$ *if and only if*:
- *each* $G_i \in 1\text{-}\mathscr{S}\mathscr{L}\mathscr{I}\mathscr{R}\mathscr{S}_\star$;
- *the components* $G_1, \dots, G_t$ *form a path*;
- $x_{1,2}$ *has label* $|V(G_1)|$ *in the labeling of* G_1;
- $x_{t-1,t}$ *has label* 1 *in the labeling of* G_t;
- *there is a labeling of each of the graphs* G_i, $1 < i < t$, *in which* $x_{i-1,i}$, *has label* 1 *and* $x_{i,i+1}$ *has label* $|V(G_i)|$.

The next result establishes an interesting bridge between minor-taking theory and minimum dynamic cost IRS.

Theorem 55 (Bodlaender et al. [5]). *For each* $k \geqslant 1$,
- *the class* k-$\mathscr{IRS}_d$ (*and its variants*) *is closed under minor-taking in the domain of connected graphs, and hence has a linear time recognition algorithm* (*but the algorithm is unknown*);
- $G \in k$-$\mathscr{IRS}_d$ *implies that* G *has a treewidth at most* $4k$ (*the same holds for its variants*);
- $G \in k$-$\mathscr{IRS}_d$ *and* G *is planar implies that* G *has a treewidth at most* $2k+3$ (*the same holds for its variants*).

The two last results of Theorem 55 implies that the class k-$\mathscr{IRS}_d$ covers a small part of all the graphs since almost all graphs (even sparse) have a treewidth $\Theta(n)$, see [49, Theorem 5.3.2, p. 58]. Note that the converse is false because Theorem 34 shows that there are graphs of treewidth bounded by k which have unbounded compactness for the uniform link-cost model, hence for all the models.

Open Question 12
- For $G \in k$-$\mathscr{IRS}_d$, what is the complexity of finding the node-labeling $\mathscr{L}$ of G?

4.2. Hierarchy of the IRS classes

The strictness of IRS can be obtained by just removing the label $\mathscr{L}(x)$ from $\mathscr{I}(x,y)$, and therefore by increasing by one at most its compactness (or linearity compactness). Because a cyclic interval is the union of two linear intervals, we have the trivial collection of inclusions:

Theorem 56 (Folklore). *For every* $k \geqslant 1$,
- k-$\mathscr{SIRS}_\star \subseteq k$-$\mathscr{IRS}_\star \subseteq (k+1)$-$\mathscr{SIRS}_\star$.
- k-$\mathscr{SLIRS}_\star \subseteq k$-$\mathscr{LIRS}_\star \subseteq (k+1)$-$\mathscr{SLIRS}_\star$.
- k-$\mathscr{LIRS}_\star \subseteq k$-$\mathscr{IRS}_\star \subseteq (k+1)$-$\mathscr{LIRS}_\star$.
- k-$\mathscr{SLIRS}_\star \subseteq k$-$\mathscr{SIRS}_\star \subseteq (k+1)$-$\mathscr{SLIRS}_\star$.

Open Question 13
- Is 1-$\mathscr{SIRS} = 1$-$\mathscr{IRS}$?
- Same question for k-$\mathscr{SIRS}$, $k > 1$?

In the following, $X \subsetneq Y$ means that $X \subseteq Y$ and $X \neq Y$. Because in [1, 32] it is shown $K_{2,2k+1} \in (k+1)$-$\mathscr{LIRS}_d$ but not in k-$\mathscr{IRS}_d$, we have:

Theorem 57 (Frederickson and Janardan [32]). *For every* $k \geqslant 1$, k-$\mathscr{IRS}_d \subsetneq (k+1)$-$\mathscr{IRS}_d$.

Theorem 58 (Bakker et al. [1]). *For every* $k \geqslant 1$,
- k-$\mathscr{LIRS}_d \subsetneq (k+1)$-$\mathscr{L}IRS_d$.
- 1-$\mathscr{LIRS}_d \subsetneq 1$-$\mathscr{IRS}_d$.

Theorems 57 and 58 can be improved by Theorem 47.

Corollary 1. *For every* $k \geqslant 1$,

- $k\text{-}\mathcal{SLIRS}_d \subsetneq k\text{-}\mathcal{LIRS}_d \subsetneq k\text{-}\mathcal{IRS}_d$.
- $k\text{-}\mathcal{SLIRS}_d \subsetneq k\text{-}\mathcal{SIRS}_d \subsetneq k\text{-}\mathcal{IRS}_d$.

Proof. From Theorem 47, for every $k \geqslant 1$, $K_{2k+1} \in \lfloor (2k+1)/2 \rfloor\text{-}\mathcal{LIRS}_d = k\text{-}\mathcal{LIRS}_d$, and also $K_{2k+1} \in k\text{-}\mathcal{SIRS}_d$. From the second point of Theorem 47, $K_{2k+1} \in t\text{-}\mathcal{SLIRS}_d$ implies that $t \geqslant \lceil (2k+1)/2 \rceil = k+1$. Hence $K_{2k+1} \notin k\text{-}\mathcal{SLIRS}_d$. It follows that $k\text{-}\mathcal{SLIRS}_d \neq k\text{-}\mathcal{LIRS}_d$ and $k\text{-}\mathcal{SLIRS}_d \neq k\text{-}\mathcal{SIRS}_d$. Similarly $K_{2k+2} \in k\text{-}\mathcal{IRS}_d$, but $K_{2k+2} \notin k\text{-}\mathcal{LIRS}_d$, and $K_{2k+2} \notin k\text{-}\mathcal{SIRS}_d$. □

4.3. Non-deterministic interval routing schemes

The study of *non-deterministic* IRS has been suggested in [77]. It consists in relaxing Condition 2b in Definition 1 in Section 2.1, a destination may belong to several arc-labels of a given node. In this case, the routing process can choose at random, or following some static rules, a link to forward the message. This allows more flexible routing for dynamic traffic of the network. An IRS in which the arc-labels overlap is so-called *multipath* IRS. An *all-shortest paths* IRS denotes an multipath IRS that encodes in every source node u all the possible shortest paths from u. Multipath IRS has been studied in [21] for chordal rings. Whereas it is NP-complete to know whether a graph supports or not a shortest path 1-LIRS (cf. Section 2.6), we have the following characterizations for all-shortest paths 1-LIRS classes.

Theorem 59 (Flammini et al. [20])

- *A graph supports an all-shortest paths* 1-*SLIRS if and only if it is a chain of complete graphs.*
- *A graph supports an all-shortest paths* 1-*LIRS if and only if it is a chain of* K_n-*star* (*see definition before Theorem* 50 *in Section* 4.1) *or* Δ_n-*ears* (*the union of a set of* n K_3 *sharing an edge* $\{x, y\}$ *of the chain, and of a* K_2-*star of base* $\{x, y\}$).

The characterization remains open for IRS of compactness $k \geqslant 1$. However, restricted to the dynamic cost model, we have the following characterization of all-shortest paths k-IRS:

Theorem 60 (Flammini et al. [20]). *For every integer* $k \geqslant 1$, *for every real* $s \geqslant 1$, *a graph* G *supports an all-shortest paths minimum dynamic cost* k-*IRS if and only if* G *supports a shortest path minimum dynamic cost* k-*IRS of stretch factor at most* s (*the same holds for the linear and/or strict variants*).

The characterization results of shortest path minimum dynamic cost k-IRS is given in Section 4.1.

Multipath k-IRS corresponds to the model of *multi-dimensional* IRS of dimension 1 (MIRS for short). This method is a natural generalization of Interval Routing, and it has been proposed by [19]. Roughly speaking, the labeling of the node is a d-tuple of integers considered as a point of an Euclidien space of dimension d. The arc-labeling consists of an union of Cartesian product of intervals $[a,b]$ (i.e., consecutive set of integers). Each product can be seen as a rectangle volume of points in a d-dimensional Euclidien space. The router sends a message along an incident link that is labeled by a rectangle volume that contains the destination point. Some results can be found in [65] (see also Theorem 63 in Section 4.5), and in [50] for deadlock-free MIRS (cf. 4.4).

Several labeling schemes based also on multipath Interval Routing are proposed in [35]. These schemes, called Topological and Hyper-Rectangle Routing Schemes, are compared with Interval Routing for several classes of graphs.

In [81], a particular case of multipath IRS called 2-adaptive IRS has been proposed. Every destination must appear in exactly two different sets $\mathscr{I}(e)$. The two choices achieves two routing paths: a shortest path and a deflecting path. Surprisingly, this potentiality decreases the compactness of the routing schemes in many cases.

As remarked in [40], the lower bounds of the compactness of all-shortest paths IRS are the same as the ones expressed in Theorems 16, 17, 31, and 34 because the proofs are based on the uniqueness of the shortest paths. These bounds also hold for shortest and multipath IRS in which each destination appears arbitrary but fixed many times in the arc-labels of every source node. Whereas they also apply for the multi-dimensional model (dimension 1), they do not apply for the 2-adaptive IRS with deflecting path developed by [81] because the deflecting path is not necessary a shortest path.

4.4. Deadlock-free interval routing schemes

The study of the trade-off between the length of the routing paths and compactness, and more generally the memory requirement, is complex and is the heart of much active research [29, 33, 63]. There are still many interesting problems related to Interval Routing which have not been solved.

Another interesting aspect related to the routing problem is the quality of the traffic of the messages and the congestion (or load of the links). Moreover problems related to deadlock often arise. A *deadlock* refers to a situation in which a set of messages is blocked forever because each packet in the set holds some resources (links or processors) that are also needed by another packet. Deadlock-free routing is relevant in the framework of *wormhole* protocols [9]. The first studies dealing with deadlock-free Interval Routing have been done in [66, 80], and in [82]. An IRS on G is *deadlock-free* if its induced routing function on G is deadlock-free (see [9]).

We can distinguish two approaches:
- study of k-IRS that induces deadlock-free routing function;
- study of k-IRS with some control mechanisms, in each router, for buffer management: A set of buffers is assigned to each router (eventually to each link) to prevent deadlock situations, e.g., a buffer on each *virtual channel* can be used (see [9]).

[80] showed results on deadlock-free IRS for 2D-grids with faulty regions. For the second approach, [82] unify the notion of virtual channel and interval assigned to links. For each edge e labeled by the intervals $[a_1, b_1] \cup \cdots \cup [a_k, b_k]$, and for each interval $[a_i, b_i]$, $2 \leqslant i \leqslant k$, it is associated a virtual channel (using the definitions given by [9]). In [82] it is showed that shortest path deadlock-free SLIRS are closed under Cartesian product.

Conjecture 4 (Zerrouk et al. [84]). *Every graph that supports a shortest path* 1-*SLIRS supports also a deadlock-free shortest path* 1-*SLIRS.*

However this approach imposes the use of $k-1$ virtual channels if k intervals per edge are used, whereas the number of buffers could be lower than this bound if a better control of the buffers is done.

In [17] IRS deadlock-free for some classic topologies such that trees, rings, grids, complete graphs and chordal rings are constructed with specific controllers. Some trade-offs are derived between the compactness of the IRS and the number of buffers in order to guarantee deadlock-free IRS for Cartesian product based graphs: hypercube, grids, and tori. The number of buffers used by the router is taken into account. The notion of buffered deadlock-free k-IRS is introduced. It is based on the notion of *acyclic orientation covering*. An acyclic orientation of a graph is just an orientation of its edges which introduces no cycles. Given an IRS R on a graph G, the number of buffers, s, used in the router can be upper bounded by the size of the smallest set of acyclic orientations of G that covers all the routing paths induced by R (see [71, Chapter 5]). In other words, there exists a set of s acyclic orientations of G such that all the arcs of every routing path of R is contained in at least one acyclic orientation of G. Note that $s \geqslant 2$.

In the following an s-buffered deadlock-free k-IRS is denoted by $\langle k, s\rangle$-IRS. Here we give only a part of the results of [17]:

Theorem 61 (Flammini [17])
- *There exist $\langle 1,2\rangle$-IRS for trees and complete graphs, $\langle 1,3\rangle$-IRS for rings, $\langle 1,2\rangle$-LIRS for grids, $\langle 2,5\rangle$-LIRS for tori.*
- *There exists a shortest path $\langle k, D+1\rangle$-IRS for graphs of diameter D and of compactness k.*
- *There exists a shortest path $\langle 1, \lceil d/2\rceil + 1\rangle$-LIRS for d-dimensional grids, and a shortest path $\langle 2, 2d+1\rangle$-LIRS for d-dimensional tori.*

In [50], interesting results about deadlock-free multi-dimensional IRS are mentioned (cf. Section 4.3). In particular some trade-offs between the number of buffers, the compactness, and the dimension are given for Butterfly, Cube Connected Cycles, Hypercube and Torus. (Recall that IRS corresponds to MIRS of dimension 1.) Finally, we would like to mention that [69] proposed deadlock-free IRS with experimental results for augmented grids and augmented tori.

4.5. Congestion of interval routing

The *edge-congestion* of a routing function R is the maximum, taken over all edges, of the number of routing paths using the same edge. Similarly, the *arc-congestion* of R is the maximum number of routing paths using the same arc. The *edge-forwarding index* (respectively *arc-forwarding index*) of G, denoted by $\pi(G)$ (respectively $\vec{\pi}(G)$), is the minimum, over all the routing functions R on G, of the edge-congestion (respectively arc-congestion) of R. Clearly, for every G, $\pi(G) \leqslant 2\vec{\pi}(G)$. See [46] for an introduction. The edge/arc-congestion of an IRS is the edge/arc-congestion of its induced routing function. The first study of congestion of Interval Routing appears in [8] where they show several results for specific topologies like trees, d-dimensional grids, and chordal rings.

Thoerem 62 (Cicerone et al. [8])
- *For each fixed $k \geqslant 1$, the problem of determining whether a graph supports a k-IRS (or its variants) of arc-congestion less than C is NP-complete.*
- *For every $n \times n$-tori T_n there exists an 1-IRS of arc-congestion at most $3n^3/20 + o(n^3) \approx 0.15n^3$. Note that $\vec{\pi}(T_n) \approx 0.125n^3$.*

Congestion of Interval Routing is treated also in [65] for arbitrary graphs.

Theorem 63 (Ružička and Štefankovič [65])
- *For every n-node graph G of maximum degree Δ and for every integer s, $1 \leqslant s \leqslant n$, there exists a multipath k-IRS on G such that its edge-congestion is at most $\pi(G) + n\Delta s$ and $k \leqslant 2 + \lceil n/(2s) \rceil$.*
- *For every n-node planar graph G of bounded degree there exists a multipath k-IRS on G such that its edge-congestion is $O(\pi(G))$ and $k = O(\sqrt{n})$.*

4.6. Interval routing for distributed problems

The general question "how much a labeling can help in the solution of some distributed problems" was posed by [27] and then further studied by [26, 28] in the framework of *Sense of Direction*.

Recently, David Peleg asked the question in [70]:

> In many contexts and many areas, once a good representation is constructed, it is often useful for more than one application. Yet in the particular case of Interval Routing, IRS representations seem to be used only for routing. Given that we have invested all this time and efforts in constructing them, it should be nice if we could use them (once constructed) to solve other problems more efficiently, in addition to routing.

Originally the investigation of Interval Routing came from distributed computing in asynchronous networks. [78] proved that minimum spanning tree, leader-election (when the node-labeling is not restricted to $\{1, \ldots, n\}$), and other related distributed problems can be solved in a ring labeled with a 1-IRS by exchanging $O(n)$ messages only, and $O(|E|+n)$ messages for arbitrary graphs, whereas in the general case $O(n\log n)$ messages for the ring, and $O(|E|+n\log n)$ messages for arbitrary graph are required [34].

More precisely, David Peleg posed in [70] the question:
is it possible to broadcast by exchanging a total of $O(n)$ messages for any graph supporting a shortest path 1-IRS? (adding $O(\log n)$ extra bits to the message)

The general problem is still open but there are some partial results in the case of all-shortest paths k-IRS (recall that an all-shortest path IRS is a multipath IRS that encodes all the shortest paths from every source node, cf. Section 4.3). In particular [16] shows that:

Theorem 64 (Eilam et al. [16]). *For every all-shortest paths* 1-*IRS on a graph* G, *and for every integer* x, *the distance in* G *between the nodes labeled* x *and* $x+1$ *is at most* 3.

It implies that the algorithm which consists in sending first the message to destination 1, then, after reception, to destination 2, and so on until the node labeled n, turns out a total of at most $3n$ messages. It is interesting to remark that if G has a full-adaptive 1-IRS of stretch factor s, then by a simulation of the previous algorithm, G has a $O(sn)$ messages broadcast algorithm.

Only partial results are known for (single) shortest path 1-IRS. In particular we have the property:

Theorem 65 (Gavoille and Mans [39]). *For every shortest path* 1-*SLIRS on an* n-*node graph* G, *and for all integers* x *and* y *such* $1\leqslant x<y\leqslant n$, *the distance in* G *between the nodes labeled* x *and* y *is at most* $\min\{y-1, n-x\}$.

It follows that nodes 1 and 2, and by symmetry the nodes $n-1$ and n, are neighbors. Because the grid with its usual shortest path 1-SLIRS labeling, the bound of Theorem 65 is tight.

Other broadcast algorithms have been proposed in [10] leading also to a linear time solution: The SSR-CAST algorithm. In the following, R is an IRS, s the source node, and M is the message to broadcast.

Algorithm (De la Torre et al. [10]). SSR-CAST(R,s,M)
- *Initialization*: s sends the message M and the set $I(s,u)$ to each of its neighbors u.
- *Forwarding step for* $v\neq s$: Upon receipt of the message M and a set I from some neighbor u, the node v identifies each neighbor $w\neq u$ such that $I\cap I(v,w)\neq\emptyset$, and then forwards to w the message M along with the set $I\cap I(v,w)$.

It turns out an optimal solution, i.e., $n-1$ messages and a time D, for graphs that admit an all-shortest paths 1-IRS. Note that the SSR-CAST algorithm must send per edge up to $4\log n$ extra bits added to the message to represent the set $I\cap I(v,w)$. Indeed, in [10], it is proved that $I\cap I(v,w)$ is composed of at most two intervals. The same performances hold for graphs that admit a shortest path 1-IRS such that from each node the routing paths induced by the 1-IRS form a tree. Such IRS are denoted $\mathrm{IRS}^{\mathrm{tree}}$. Note that the same algorithm can be used to compute a minimum spanning tree of the graph.

Open Question 14
- Is $1\text{-IRS} = 1\text{-IRS}^{\mathrm{tree}}$?
- Same question for k-IRS and its variants, $k > 1$?

To conclude, we think that Interval Routing is certainly a sufficiently simple model for the theoretical study of the structural and implicit information of distributed networks. It is quite easy to predict that this model will be extended and/or exploited in many fashions. This is what seems to be the thesis of the paper of [80].

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