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(na začátku vynechávám 4 kapitoly, aby bylo číslování stejné, jako v přednášce)

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CHAPTER 5

Primitivni funkce

5.0.1. motivace. populace: $f\prime = c * f$
zname: $\exp x$, ale nevime, ze je jedina, v(0) je N_0

5.1. zakladni vlastnosti primitivnich funkci

DEFINITION 5.1.1. **(klicovy pojem)** Necht funkce F je definovana na otevrenem intervalu $(a, b)\,; a < b \in \mathbf{R}^*$. Necht F ma v kazdem bode $x \in (a, b)$ vlastni derivaci $F\prime(x) = f(x)$. Pak rikame, ze funkce F je **primitivni funkci** k funkci f.

5.1.1. jednoznacnost. $f \to f\prime$ je jednoznacna operace (limita je jednoznacna)
$f\prime \to f$ neni jednoznacna operace:
$f == x \Rightarrow f\prime = 0 \Rightarrow (g(x) + c)\prime = g\prime(x)$ - minimalne pricist konstantu

THEOREM. *1. (jednoznacnost integralu)*
Necht F, G jsou primitivni funkce funkce f na otevrenem intervalu I. Pak $\exists c \in \mathbf{R}$ takove, ze $F(x) = G(x) + c$ MA1

PROOF. Polozme $H(x) = F(x) - G(x)$.
Potom $\forall x \in I : H\prime(x) = F\prime(x) - G\prime(x) = f(x) - f(x) = 0$
$\Rightarrow H$ je na I konstantni □

NOTE 5.1.2. (i) f ma primitivni funkci F. Potom F je spojita (nebot $\forall x \in I : \exists F\prime(x)$)
(ii) $f(x) = sign(x)$ nema primitivni funkci - dukaz nasledujici vety
(iii) priklad: $f(x) = \begin{matrix} x^2 * \sin\left(\frac{1}{x^2}\right) & x \neq 0 \\ 0 & x = 0 \end{matrix}$ MA2
$x \neq 0 : f\prime(x) = 2x \sin\left(\frac{1}{x^2}\right) + x^2 \cos\left(\frac{1}{x^2}\right)(-2)\frac{1}{x}$
$x = 0 : f\prime(0) = \lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = \frac{x^2 * \sin\left(\frac{1}{x^2}\right)}{x} = \lim_{x\to 0} x * \sin\left(\frac{1}{x^2}\right) = 0 \; (0 * omezena)$
derivace existuje vsude, ale v 0 neni spojita.
tedy: $f\prime$ ma primitivni funkci (f), ale neni spojita
(iv) casem dokazeme: f spojita $\Rightarrow f$ ma primitivni funkci $\Rightarrow f$ ma Darbouxovu vlastnost (nabyva vsech mezihodnot)

THEOREM. *2.*
Necht f je spojita na otevrenem intervalu I. Pak f ma na I primitivni funkci.

PROOF. (zatim bez dukazu) bude ... □

NOTE 5.1.3. vsechny dukazy se delaj tak, ze se to prevede na derivovani

THEOREM. *3. (linearita primitivnich funkci)*
Necht f ma na otevrenem intervalu I primitivni funkci F.
Necht g ma na I (stejny interval) primitivni funkci G.
Necht $\alpha, \beta \in \mathbf{R}$
Pak $\alpha f + \beta g$ ma primitivni fukci $\alpha F + \beta G$.

PROOF. $(\alpha F + \beta G)\prime = \alpha F\prime + \beta G\prime = \alpha f + \beta g$ □

znaceni: F primitivni funkce k f ($\Leftrightarrow F\prime = f$). $F(x) = \int f(x)\,dx$ ... mnozina vsech primitivnich funkci k funkci f, tedy $\int f(x)\,dx = F(x) + c$, kde $c \in \mathbf{R}$
(nekdy se znaci $\overset{c}{=}$ - az na konstantu)
$\int$ ...suma po kousickach

5.1.2. tabulkove integraly.

$f(x)$	$\int -c$	$x \in$		
$n^n, n \in \mathbf{Z} \setminus \{-1\}$	$\frac{x^{n+1}}{n+1}$	$x \in \mathbf{R}$ pokud $n \geq 0$,$x \in \mathbf{R} \setminus \{0\}$ pokud $n < 0$		
x^{-1}	$\log\|x\|$	$x \neq 0$		
e^x	e^x	$\mathbf{R}$		
$\sin x$	$-\cos x$	$\mathbf{R}$		
$\cos x$	$\sin x$	$\mathbf{R}$		
$\frac{1}{\cos^2 x}$	$\tan x$	$\left(-\frac{\pi}{2}; \frac{\pi}{2}\right) + k\pi, k \in \mathbf{Z}$		
$\frac{1}{\sin^2 x}$	$-cotg(x)$	$(0; \pi) + k\pi, k \in \mathbf{Z}$		
$\frac{1}{1+x^2}$	$\arctan x$	$\mathbf{R}$		
$\frac{1}{\sqrt{1-x^2}}$	$\arcsin x = -\arccos x$	$(-1, 1)$		

Zatim neumime: $\int \log x dx$, $\cos^2 x$, ...

Theorem. *4. (Darbouxova vlastnost derivace) LV*
Necht f ma na otevrenem I primitivni funkci. Pak f ma na I Darbouxovu vlastnosti, tj.
$\forall y_1 < y_2 \in f(I); \forall z \in (y_1, y_2) : \exists x \in I$ *tak, ze* $f(x) = z$

Proof. /*sikovna funkce $H(x) = F(x - xz)$ a derivace a tak*/
$y_1 < y_2 \in f(I), y_1 = f(x_1), y_2 = f(x_2)$
BUNO: $x_1 < x_2$
Vezmu $z \in (y_1, y_2)$
Poloz: $H(x) = F(x) - xz$ kde $F(x) = \int f(x)\,dx$ (F je nektera primitivni funkce k f)
Potom H je spojita na I a ma vlastni derivaci $H\prime(x) = f(x) - z \forall x \in I$
H je spojita na $[x_1, x_2] \Rightarrow H$ nabyva na $[x_1, x_2]$ sveho minima.
- v uvahu pripadaj krajni, derivace (0, neex)
$H\prime(x_1) = f(x_1) - z = y_1 - z < 0 \Rightarrow H$ nema minimum v x_1
stejne $H\prime(x_2) = f(x_2) - z = y_2 - z > 0 \Rightarrow H$ nema minimum v x_2
Tedy $\exists x \in (x_1, x_2)$ takove, ze H ma minimum v $x \Rightarrow H\prime(x) = 0 \Rightarrow f(x) - z = 0$ □

5.1.3. metody per partes, substituce.

Theorem. *5. (1. o substituci) LV*
Necht f ma primitivni funkci F na (a, b).
Necht φ je definovana na intervalu (α, β), necht φ ma hodnoty v (a, b)
Necht φ ma v kazdem bode $t \in (\alpha, \beta)$ vlastni derivaci $\varphi\prime(t)$.
Potom plati:
$\int f(\varphi(t))\,\varphi\prime(t)\,dt = F(\varphi(t)) + c, t \in (\alpha, \beta)$

Proof. zderivujem pravou stranu
$(F(\varphi(t)) + c)\prime = F\prime(\varphi(t))\,\varphi\prime(t) = f(\varphi(t))\,\varphi\prime(t)$ □

Note 5.1.4. velmi uzitecna pro integrovani linearnich funkci
Necht f ma primitivni funkci F (na (a, b))
Pak $f(\alpha x + \beta), \alpha \neq 0$ ma primitivni funkci $\frac{F(\alpha x+\beta)}{\alpha}$

Proof. Zvolim $\varphi(t) = \alpha t + \beta$
dle V5 $\int f(\alpha t + \beta)\,\alpha dt = F(\alpha t + \beta) + c \;/: \alpha$
$\int f(\alpha t + \beta)\,dt = \frac{F(\alpha t+\beta)}{\alpha} + c$ □

Example 5.1.5. $\int \sin(2x + 1)\,dx = \frac{-\cos(2x+1)}{2} + c$
$\int \frac{dx}{\sqrt{3x+4}} = \frac{2\sqrt{3x+4}}{3} + c$

Theorem. *6. (2. o substituci) LV*
Necht φ je definovana na intervalu (α, β), necht φ ma v kazdem bode $t \in (\alpha, \beta)$ ***nenulovou***[1] *vlastni derivaci*
Necht $\int f(\varphi(t))\,\varphi\prime(t)\,dt = G(t) + c, t \in (\alpha, \beta)$, kde f je definovana na (a, b), $(a, b) = \varphi(\alpha, \beta)$.
Potom plati: $\int f(x)\,dx = G\left(\varphi^{-1}(x)\right) + c, x \in (a, b)$

Proof. /*OK*//* $\varphi\prime$ ma primitivni funkci na $(\alpha, \beta) \overset{V4}{\Rightarrow}$ ma Darbouxovu vlastnost $\Rightarrow \varphi\prime$ nemeni znamenko na $(\alpha, \beta) \Rightarrow$ je ryze monotonni a spojita $\Rightarrow \exists \varphi^{-1}(x)$ na (a, b) a lze pouzit V4.19 */
$\left[G\left(\varphi^{-1}(x)\right)\right]\prime \overset{V4.18}{=} G\prime\left(\varphi^{-1}(x)\right)\left(\varphi^{-1}\right)\prime(x) \overset{predpoklad, V4.19}{=} f\left(\varphi\left(\varphi^{-1}(x)\right)\right)\varphi\prime\left(\varphi^{-1}(x)\right)\frac{1}{\varphi\prime(\varphi^{-1}(x))} = f(x)$
$???\left(\varphi^{-1}\right)\prime(x) = \frac{1}{\varphi\prime(x)}$ □

[1] potrebujeme, aby byla ryze monotonni

Example 5.1.6. pouziti:
najdu vhodnou φ. $x=\varphi(t)$, umim spocitat $\int f(\varphi(t))\varphi\prime(t)\,dt=G(t)$
dosadim $t=\varphi^{-1}(x)\Rightarrow\int f(x)\,dx=G\left(\varphi^{-1}(x)\right)+c$
$\int\sqrt{1-x^2}dx, x\in(-1,1)$ - kruznice
zvolme $\varphi(t)=\sin t$. pak: $\varphi\prime(t)=\cos t,\alpha=-\frac{\pi}{2},\beta=\frac{\pi}{2}$
$\Rightarrow\int f(\varphi(t))\varphi\prime(t)\,dt=\int\sqrt{1-(\sin t)^2}\cos t dt=\int|\cos t|\cos t dt=\int\cos^2 t dt$
vzorec: $\cos^2 t=\frac{1+\cos(2t)}{2}\Rightarrow\int\cos^2 t dt=\int\frac{1+\cos(2t)}{2}dt=\frac{1}{2}\int dt+\frac{1}{2}\int\cos(2t)\,dt=\frac{t}{2}+\frac{1}{2}\frac{\sin(2t)}{2}+c$
V6: dosadime za $t=\varphi^{-1}(x)=\arcsin x, t\in(-1,1)$
takze $\int\sqrt{1-x^2}dx=\frac{\arcsin x}{2}+\frac{1}{2}\frac{\sin(2\arcsin x)}{2}+c$

Note 5.1.7. v trigonometrickejch se mocniny prevadi na nasobek argumentu

pomucka: $\int f(x)\,dx, x\in(a,b)$
najdu $\varphi:\varphi(\alpha,\beta)=(a,b),\exists\varphi\prime\neq 0$ na (α,β)
dosazuji: $x=\varphi(t)$
$dx=\varphi\prime(t)\,dt$ (lze chapat jako $\frac{dx}{dt}=\varphi\prime(t),(x=\varphi(t)))$
pak vyjde $\int f(x)\,dx=\int f(\varphi(x))\varphi\prime(t)\,dt=G(t)+c=G\left(\varphi^{-1}(x)\right)+c$
dosad: $t=\varphi^{-1}(x)$

Theorem. *7. (integrace per partes) LV*
Necht f,g jsou 2 spojite funkce na (a,b), necht $F\prime=f$ a $G\prime=g$ a (a,b)
Potom
$\int g(x)F(x)\,dx=F(x)G(x)-\int G(x)f(x)\,dx, x\in(a,b)$

Proof. $(F(x)G(x))\prime=f(x)G(x)+F(x)g(x)\;/\int dx$
$F(x)G(x)=\int(f(x)G(x)+g(x)F(x))\,dx=\int(f(x)G(x))\,dx+\int(g(x)F(x))\,dx$ (nebot fG,Fg spojite a tedy primitivni funkce $\exists$) □

Note 5.1.8. predpoklad spojitosti tam byt nemusi

Example 5.1.9. $\int\log x dx, x\in(0,\infty)$
$=\int 1.\log(x)\;f(x)=1,G(x)=\log x$
$=x\log x-\int x\frac{1}{x}=x(\log x-1)+c$

-

Example 5.1.10. $\int e^x\sin(x)\,dx\;f(x)=e^x,G(x)=\sin x$
$=e^x\sin x-\int e^x\cos x dx=$
$f(x)=e^x,G(x)=\cos x$
$=e^x\sin x-e^x\cos x-\int e^x\sin x dx$
$\Rightarrow 2\int e^x\sin(x)\,dx=e^x\frac{(\sin x-\cos x)}{2}+c$

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Example 5.1.11. Spoctete $\int\frac{dx}{(1+x^2)^n}, n\in\mathbf{N}, x\in\mathcal{R}$
oznacme $I_n=\int\frac{dx}{(1+x^2)^n}$
$I_1=\arctan x+c$
$I_n=\int 1*\frac{dx}{(1+x^2)^n}=\frac{x}{(1+x^2)^n}-\int x(-n)\frac{2x}{(1+x^2)^{n+1}}dx=\frac{x}{(1+x^2)^n}+2n\int\frac{x^2}{(1+x^2)^{n+1}}dx$
trik: pucim si 1
$=\frac{x}{(1+x^2)^n}+2n\int\frac{1+x^2-1}{(1+x^2)^{n+1}}dx=\frac{x}{(1+x^2)^n}+2n\int\frac{dx}{(1+x^2)^n}-\int\frac{1}{(1+x^2)^n}dx$
$=\frac{x}{(1+x^2)^n}+2nI_n-2nI_{n+1}$
$\Rightarrow I_n(1-2n)=\frac{x}{(1+x^2)^n}-2nI_{n+1}$
$I_{n+1}=\frac{1}{2n}\frac{x}{(1+x^2)^n}+\left(1-\frac{1}{2n}\right)I_n$

5.2. integrace racionalnich funkci

Definition 5.2.1. **Racionalni funkci** rozumime funkci $R(x)=\frac{P(x)}{Q(x)}, x\in\mathbf{R}$, kde P,Q jsou polynomy a $Q\neq\neq 0$

Theorem. *(zakladni veta algebry)*
(bez dukazu)
Necht P je polynom na $\mathbf{R}$, $\deg P=n\in\mathbf{N}$
Potom $\exists x_1,\ldots,x_n\in\mathbf{C}:P(x)=a_n(x-x_1)\ldots(x-x_n)$
Pro realne funkce $\exists$ v $\mathbf{R}$ nerozlozitelne kvadraticke trojcleny (x^2+px+q, kde $p^2<4q$ - parabola je nad nebo pod osou)

motivace: chteli bysme:
$\frac{P(x)}{Q(x)}=\frac{P(x)}{a_n(x-x_1)\ldots(x-x_n)}=\text{SNAD}=\left(\frac{A}{x-x_1}+\ldots+\frac{Z}{x-x_n}\right)\frac{1}{a_n}$
avsak muze se stat:
$=\frac{Ax+C}{x^2+px+q}+\frac{Bx+D}{x^2+rx+s}+\ldots$
zakladni ingredience:

$A, a \in \mathbf{R}, \int \frac{A}{x-a}dx = A\log|x-a| + c, x \in (-\infty, a) \vee (a, \infty)$.

$\frac{A}{(x-a)^n}$. $\int \frac{A}{(x-a)^n}dx = \begin{cases} A\log|x-a|, x \neq a, n = 1 \\ \frac{A}{1-n}\frac{1}{(x-a)^{n-1}}, x \neq a, n > 1 \end{cases}$

$A, B, p, q \in \mathbf{R}, \int \frac{Ax+B}{x^2+px+q}dx$. $I = \frac{Ax+B}{(x^2+px+q)^n}dx$ - rozdelim na dva - nahore typ "vidim derivaci" + konstanta$\int \frac{1}{blagh}$ ¯prevedeme na arctan

$= \frac{A}{2}\underbrace{\int \frac{2x+p}{(x^2+px+q)^n}dx}_{I_1} + \left(B - \frac{Ap}{2}\right)\underbrace{\int \frac{1}{(x^2+px+q)^n}dx}_{I_2}$

I_1: subst $y = x^2+px+q \to \int \frac{dy}{y^n} = \begin{cases} \log|y| + c, y \neq 0, n = 1 \\ \frac{1}{1-n}\frac{1}{y^{n-1}} + c, y \neq 0, n > 1 \end{cases}$

$\Rightarrow I_1 = \begin{cases} \log\left(x^2+px+q\right), y \neq 0, n = 1 \\ \frac{1}{1-n}\frac{1}{(x^2+px+q)^{n-1}}, y \neq 0, n > 1 \end{cases}$

Note 5.2.2. Definovanost na celem R - ireducibilita trojclenu

$I_2 = \int \frac{1}{(x^2+px+q)^n}dx = \int \frac{1}{\left(\left(x+\frac{p}{2}\right)^2 + q - \frac{p^2}{4}\right)^n}$

$q - \frac{p^2}{4} > 0$ - plyne z ireducibility trojclenu

$= \frac{1}{\left(q-\frac{p^2}{4}\right)^n}\int \frac{1}{\left(\left(\frac{x+\frac{p}{2}}{\sqrt{q-\frac{p^2}{4}}}\right)^2+1\right)^n}dx$

subst $y = \left(\frac{x+\frac{p}{2}}{\sqrt{q-\frac{p^2}{4}}}\right)$

pak $dx = \sqrt{q-\frac{p^2}{4}}dy$

a tedy $I_2 = \frac{1}{\left(q-\frac{p^2}{4}\right)^{n-\frac{1}{2}}}\int \frac{dy}{(1+y^2)^n}$ I_n z minule prednasky

Example 5.2.3. pro $n = 1$: $I_2 = \frac{1}{\sqrt{q-\frac{p^2}{4}}}\arctan\left(\frac{x+\frac{p}{2}}{\sqrt{q-\frac{p^2}{4}}}\right) + c$

Theorem. *8. (o rozkladu na parcialni zlomky) (bez dukazu)*
Necht P, Q jsou polynomy s realnymi koeficienty a neplati $Q \equiv 0$
(i) $\deg P < \deg Q$
(ii) $Q(x) = a_n(x-x_1)^{\alpha_1}\dots(x-x_k)^{\alpha_k}\left(x^2+p_1x+q_1\right)^{\beta_1}\dots\left(x^2+p_lx+q_l\right)^{\beta_l}\ \forall x \in \mathbf{R}$
(iii) $a_n, x_1\dots x_k, , p_1, q_1\dots p_l, q_l \in \mathbf{R}, a_n \neq 0$
(iv) $\alpha_1, \dots, \alpha_k, \beta_1, \dots \beta_l \in \mathbf{N}$
(v) $x^2+p_ix+q_i$ *je ireducibilni* $\forall i = 1, \dots, l$
(vi) $x_i \neq x_j \forall i \neq j = 1, \dots, k$
Potom $\exists$ jednoznacne urcena cisla $A_1^1, \dots, A_{\alpha_1}^1, \dots, A_1^k, \dots, A_{\alpha_k}^k, B_1^1, C_1^1, \dots, B_{\beta_1}^1 C_{\beta_1}^1, \dots, B_1^l, C_1^l, \dots, B_{\beta_l}^l C_{\beta_l}^l$
tak, ze $\frac{P(x)}{Q(x)} = \frac{A_1^1}{x-x_1} + \dots + \frac{A_{\alpha_1}^1}{(x-x_1)^{\alpha_1}} + \dots + \frac{A_1^k}{x-x_k} + \dots + \frac{A_{\alpha_k}^k}{(x-x_k)^{\alpha_k}}$
$+\frac{B_1^1x+C_1^1}{x^2+p_1x+q_1} + \dots + \frac{B_{\beta_1}^1+C_{\beta_1}^1}{(x^2+p_1x+q_1)^{\beta_1}} + \dots + \frac{B_1^lx+C_1^l}{x^2+p_lx+q_l} + \dots + \frac{B_{\beta_l}^l+C_{\beta_l}^l}{(x^2+p_lx+q_l)^{\beta_l}}$

Lemma. *Jestlize P, Q jsou polynomy s realnymi koeficienty, Pak $P \equiv Q \Leftrightarrow P, Q$ maji stejne koeficienty*

Proof. $P \equiv Q \Leftrightarrow P - Q \equiv 0$
Tedy staci dokazat: S je nulovy polynom $\Leftrightarrow \forall$ koefifienty S jsou nulove
"$\Leftarrow$" zrejme
"$\Rightarrow$" S nulovy $\Rightarrow S$ ma ∞ korenu $\Rightarrow \deg S = -1 \Leftrightarrow \forall$koeficienty$S = 0$
(kdyby mel n korenu, byl by stupen n) □

Summary 5.2.4. Postup pri integraci racionalnich funkci:

(1) $\int \frac{P(x)}{Q(x)}dx = \int P_1(x)dx + \int \frac{P_2(x)}{Q(x)}dx, \deg P_2 < \deg Q$ delenim mnohoclenu
(2) $\int P_1(x)dx = I_1$ OK
(3) $\int \frac{P_2(x)}{Q(x)}dx = I_2$ rozlozim zlomek na parciani zlomky podle V8
(4) urcime koeficienty A, B, C pomoci lemmatu
(5) integrujeme vznikle parcialni zlomky

Example 5.2.5. Najdete $I = \int \frac{x^2+1}{(x+1)(x^2+x+1)}dx$
(3) $\frac{x^2+1}{(x+1)(x^2+x+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+x+1}\ /(x+1)\left(x^2+x+1\right)$
(4) $x^2+1 = \left(x^2+x+1\right)A + (x+1)(Bx+C)$
zvolim $x = -1$ $2 = A$
$\Rightarrow -x^2-2x-1 = Bx^2 + (B+C)x + C \Rightarrow B = C = -1$
$\frac{x^2+1}{(x+1)(x^2+x+1)} = \frac{2}{x+1} - \frac{x+1}{x^2+x+1}$

$\Rightarrow I = 2\underbrace{\int \frac{dx}{x+1}}_{I_1} - \underbrace{\int \frac{x+1}{x^2+x+1}}_{I_2}$

$I_1 = \log|x+1| + c, x \neq -1$

I_2 =potrebuju nahoru derivaci spodku $= \frac{1}{2}\underbrace{\int \frac{2x+1}{x^2+x+1}dx}_{I_3} + \frac{1}{2}\underbrace{\int \frac{1}{x^2+x+1}dx}_{I_4}$

$I_3 = \log(x_2 + x + 1) + c, x \in \mathbf{R}$

$I_4 = \int \frac{1}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}dx = \frac{4}{3}\int \frac{1}{\left(\frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)^2+1}dx$

subst $y = \frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}}$

$dx = \frac{\sqrt{3}}{2}dy$

$I_4 = \frac{2}{\sqrt{3}}\int \frac{1}{y^2+1}dy = \frac{2}{\sqrt{3}}\arctan y + c = \frac{2}{\sqrt{3}}\arctan\left(\frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right) + c$

$I = 2\log|x+1| - \frac{1}{2}\log(x_2+x+1) - \frac{1}{\sqrt{3}}\arctan\left(\frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right) + c, x \neq -1$

5.3. integrace trigonometrickych funkci

kam s timdle . .. [2]

Jak integrovat $\int \frac{1}{1+\cos x}dx$?

5.3.1. uzitecne substituce. $P(x,y), Q(x,y)$ polynomy 2 promennych

$P(x,y) = \sum_{i,j=1}^{n} a_{ij}x^i y^j$

$R(x,y) = \frac{P(x,y)}{Q(x,y)}$

$\int R(\sin x, \cos x)\,dx$

substituce. (i) $t = \tan\frac{x}{2}$ - vzdy vede na racionalni fukci

protoze: $x = 2\arctan y, dx = \frac{2}{1+y^2}dy$

$\sin x = \frac{2y}{1+y^2}, \cos x = \frac{1-y^2}{1+y^2}$

(ii) licha $y = \cos x$ pokud $R(-\sin x, \cos x) = -R(\sin x, \cos x)$

(iii) licha $y = \sin x$ pokud $R(\sin x, -\cos x) = -R(\sin x, \cos x)$

(iv) suda $y = \tan x$ pokud $R(-\sin x, -\cos x) = R(\sin x, \cos x)$

rada: (i) az po vsech ostatnich

NOTE 5.3.1. substituce nejsou vsude definovany - nehodi se na celem $\mathbf{R}$ (pro substituci musi byt funkce sprosta)

EXAMPLE 5.3.2. $I = \int \frac{dx}{1+\sin^2 x}, x \in \mathbf{R}$

funkce je suda, subst $y = \tan x, x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$y^2 = \frac{\sin^2 x}{\cos^2 x} \Rightarrow \sin^2 x = y^2\cos^2 x = y^2\left(1-\sin^2 x\right) \Rightarrow \sin^2 x = \frac{y^2}{1+y^2}$

$x = \arctan y$

$dx = \frac{1}{1+y^2}dy$

$I = \int \frac{\frac{1}{1+y^2}dy}{1+\frac{y^2}{1+y^2}} = \int \frac{dy}{1+2y^2} = \int \frac{dy}{1+\left(\sqrt{2}y\right)^2} \overset{\text{substituce 1}}{=} \frac{1}{\sqrt{2}}\arctan\left(\sqrt{2}y\right) = \frac{1}{\sqrt{2}}\arctan\left(\sqrt{2}\tan x\right) + C, x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

to same je mozne udelat na nasobcich π, ale primitivni funkce musi byt spojita MA3

"lepeni"

dejme 1. $c = 0$ (muzu zvolit)

$F(x) = \frac{1}{\sqrt{2}}\arctan\left(\sqrt{2}\tan x\right)$

$\lim_{x\to-\frac{\pi}{2}+} F(x) = -\frac{\pi}{2\sqrt{2}}$

$\lim_{x\to-\frac{\pi}{2}-} F(x) = \frac{\pi}{2\sqrt{2}}$

$\Rightarrow$ je treba graf na $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$ posunout o $\frac{\pi}{2\sqrt{2}} - \left(-\frac{\pi}{2\sqrt{2}}\right) = \frac{\pi}{\sqrt{2}}$

$$G(x) = \begin{cases} F(x) + k\frac{\pi}{\sqrt{2}}, x \in \left(\frac{\pi}{2} + k\pi, \frac{3\pi}{2} + k\pi\right), k \in \mathbf{Z} \\ \frac{\pi}{2\sqrt{2}} + k\frac{\pi}{\sqrt{2}}, x = \frac{\pi}{2} + k\pi, k \in \mathbf{Z} \end{cases}$$

---

[2] $t = \tan\frac{x}{2}, x \in (-\pi, \pi)$

$\cos x = \frac{1-t^2}{1+t^2} = \ldots = \left(\tan\frac{x}{2} = \frac{\sqrt{\frac{1-\cos\alpha}{2}}}{\sqrt{\frac{1+\cos\alpha}{2}}} = \ldots\right)$

$\sin x = \frac{2t}{1+t^2} = \ldots = \left(= \frac{\sqrt{(1+t^2)^2-(1-t^2)^2}}{1+t^2} = \frac{\sqrt{(1+t^2)^2-(1-t^2)^2}}{1+t^2}\right)$

subst $y = \tan x, x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. $y^2 = \frac{\sin^2 x}{\cos^2 x} \Rightarrow \sin^2 x = y^2\cos^2 x = y^2\left(1-\sin^2 x\right) \Rightarrow \sin^2 x = \frac{y^2}{1+y^2}$

$\cos^2 x = 1 - \frac{y^2}{1+y^2} = \frac{1}{1+y^2}$

$dy = \frac{1}{\cos^2 x}dx = \frac{1}{1-\sin^2 x}dx = \frac{1}{1-\frac{y^2}{1+y^2}}dx = 1 + y^2 dx$

$\Rightarrow dx = \frac{1}{1+y^2}dy$

$G(x) + C = I$

(1) podivat se, zda $\exists$ naka jednoducha trigonometricka uprava (e.g. $\tan^2 x = \tan^2 x + 1 - 1 = 1 + \frac{\sin^2 x}{\cos^2 x} - 1 = \frac{1}{\cos^x} - 1$)
(2) zvolit vhodnou substituci
(3) slepit

5.4. integrace funkci typu $R\left(x, \left(\frac{Ax+B}{Cx+D}\right)^{\frac{1}{q}}\right), q \in N, q > 1$

5.4.1. predpoklad: $AD \neq CB$ $\left(\left(\frac{\frac{1}{C}(CAx+CB)}{\frac{1}{A}(ACx+AD)}\right)^{\frac{1}{q}} = \left(\frac{A}{C}\right)^{\frac{1}{q}}\right)$. Vzdy substituujeme $y = \left(\frac{Ax+B}{Cx+D}\right)^{\frac{1}{q}}$
$\Rightarrow Ax + B = (Cx + D)\,y^q \Rightarrow Ax - Cxy^q = Dy^q - B \Rightarrow x = \frac{Dy^q - B}{(A - Cy^q)}$
$dx = \frac{Dqy^{q-1}(A-Cy^q)+(Dy^q-B)qCy^{q-1}}{(A-Cy^q)^2}dy = qy^{q-1}\frac{D(A-Cy^q)+C(Dy^q-B)}{(A-Cy^q)^2}dy =$
$= qy^{q-1}\frac{DA-CB}{(A-Cy^q)^2}dy$

EXAMPLE 5.4.1. $I = \int \frac{\sqrt{2x+3}+x}{\sqrt{2x+3}-x}dx$
subst $y = \sqrt{2x+3}, x \in \left(-\frac{3}{2}, \infty\right)$
$x = \frac{y^2-3}{2}$
$dx = ydy$
$I = \int \frac{y+\frac{y^2-3}{2}}{y-\frac{y^2-3}{2}}ydy = \int \frac{y^3+2y^2-3y}{-y^2+2y+3}dy$
ted uz je to racionalni
$4y^2 - 3y$
$8y + 12$
$= \int -y - 4dy + \underbrace{\int \frac{8y+12}{-y^2+2y+3}dy}_{I_1}$
$I_1 = 4\int \frac{2y+3}{(-y+3)(y+1)} = \frac{A}{y-3} + \frac{B}{y+1}$
$A = \frac{-24-12}{4} = -9$
$B = \frac{8-12}{-4} = 1$
$\Rightarrow I = \int -y - 4 - \frac{9}{y-3} + \frac{1}{y+1}dy = -\frac{y^2}{2} - 4y - 9\log|y-3| + \log|y+1| + c, y \neq -1, 3$
$I = -\frac{2x+3}{2} - 4\sqrt{2x+3} - 9\log\left|\sqrt{2x+3} - 3\right| + \log\left|\sqrt{2x+3}+1\right| + c, x = \left(-\frac{3}{2}, 3\right) \vee (3, \infty)$

5.5. Eulerovy substituce

tykaji se vyrazu $R\left(x, \sqrt{ax^2+bx+c}\right), a \neq 0$
(i) $ax^2 + bx + c$ ma dvojnasobny $\mathbf{R}$ koren $\alpha \in \mathbf{R}$ (diskriminant je nulovy)
pak $a > 0$ (jinak neni otevreny interval na integrovani)
potom $\sqrt{ax^2+bx+c} = \sqrt{a(x-\alpha)^2} = \sqrt{a}\,|x-\alpha|$ umime
(ii) $ax^2 + bx + c$ ma 2 ruzne koreny $\alpha_1 < \alpha_2$ $(ax^2 + bx + c = a(x-\alpha_1)(x-\alpha_2)$.
$a > 0$: $\sqrt{ax^2+bx+c} = \sqrt{a(x-\alpha_1)(x-\alpha_2)} = \sqrt{a(x-\alpha_1)^2\frac{x-\alpha_2}{x-\alpha_1}} = |x-\alpha_1|\sqrt{a\frac{x-\alpha_2}{x-\alpha_1}}$ - predchozi pripad $x \in (-\infty, \alpha_1) \vee x \in (\alpha_2, \infty)$
$a < 0$: $\sqrt{ax^2+bx+c} = \sqrt{-a(x-\alpha_1)(\alpha_2-x)} = \sqrt{-a(x-\alpha_1)^2\frac{\alpha_2-x}{x-\alpha_1}} = |x-\alpha_1|\sqrt{-a\frac{\alpha_2-x}{x-\alpha_1}}$ - predchozi pripad $x \in (\alpha_1, \alpha_2)$
(iii) $ax^2 + bx + c$ nema realne koreny ($\Rightarrow a > 0$) - Eulerova substituce
$\sqrt{ax^2+bx+c} = \sqrt{a}x + y$

EXAMPLE 5.5.1. $I = \int \frac{dx}{x^2\sqrt{x^2+1}}$
subst $\sqrt{x^2+1} = x + y$
$y = \sqrt{x^2+1} - x$
$x^2 + 1 = x^2 + 2xy + y^2$
$\Rightarrow x = \frac{1-y^2}{2y}$
$dx = \frac{-1-y^2}{2y^2}dy$
$\int = \frac{2}{\left(\sqrt{x^2+1}-x\right)^2-1} + x, x \neq 0$
"ten priklad se mi nak nechce dopocitavat"

CHAPTER 6

urcity integral

6.1. Riemannův integrál

Motivace: plocha 2D objektu
podminky:

(1) musi mit smysl
(2) ≥ 0
(3) $F_1 \subseteq F_2 \Rightarrow P(F_1) \leq P(F_2)$

Řekové: "method of Exhaustion"
ctverec.
obdelnik. $P = ab$
MA4
funkce konstantni. $P = k(b-a)$
schodovita funkce f MA5. $P = \sum_{j=1}^{n} k_j (x_j = x_{j-1})$
17. stoleti .. Newton. Leibniz - $\int$
19. .. Riemann, Darboux - zaklady urciteho $\int$
20. - Lebesqueův $\int$, Kurzweiův-Peronnův $\int$
Riemann. sendvič MA6
2 schodovité funkce

Definition 6.1.1. Necht $[a,b] \subset \mathbf{R}$. Konecnou posloupnost $\{x_j\}_{j=0}^{n}, n \in \mathbf{N}$ nazvu **delenim intervalu** $[a,b]$, jestlize $a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b$

Deleni intervalu oznacujeme D $(D = \{x_j\}_{j=0}^{n}, n \in \mathbf{N})$. Jestlize D_1, D_2 jsou dve deleni $[a,b]$, potom D_2 nazvu **zjemnenim** D_1, jestlize kazdy delici bod D_1 je take delicim bodem D_2.
-

Definition 6.1.2. Necht f je omezena a definovana na $[a,b] \subset \mathbf{R}$. Necht D je delenim $[a,b]$. Pak definujeme:
$s(f,D) = \sum_{j=1}^{n} \inf_{x \in [x_{j-1},x_j]} f(x)(x_j - x_{j-1})$ (**dolni soucet**)
$S(f,D) = \sum_{j=1}^{n} \sup_{x \in [x_{j-1},x_j]} f(x)(x_j - x_{j-1})$(**horni soucet**)
MA7
(klicovy pojem) $\int_{\underline{a}}^{b} f(x)\,dx = \sup_D s(f,D)$ - supremum pred vsechny mozne deleni $[a,b]$ (**dolni Riemannuv integral)**
$\int_{a}^{\overline{b}} f(x)\,dx = \inf_D S(f,D)$ - supremum pres vsechny mozne deleni $[a,b]$ (**horni Riemannuv integral)**
Jestlize $\int_{\underline{a}}^{b} f(x)\,dx \neq \int_{a}^{\overline{b}} f(x)\,dx$, pak f **neni riemannovsky integrovatelna.**

Jestlize $\int_{\underline{a}}^{b} f(x)\,dx = \int_{a}^{\overline{b}} f(x)\,dx$, pak f **je riemannovsky integrovatelna** a hodnotu $(R)\int_{a}^{b} f(x)\,dx = \int_{\underline{a}}^{b} f(x)\,dx$ nazyvame **Riemannovym integralem pres interval** $[a,b]$

Mnozinu vsech riemannovsky integrovatelnych funkci na $[a,b]$ znacime $R([a,b])$.

6.1.1. problemy.

(1) podminky integrability
(2) vlastnosti $R([a,b])$ (napr. vztahy ke spojitosi, $\exists \int$, Darbouxovosti)
(3) vlastnosti zobrazeni $f \to \int_{a}^{b} f(x)\,dx$ (na pevnem $[a,b]$, nebo v zavislosti na zmene $[a,b]$)
(4) vypocet integralu

6.1.2. umluva ("karlínská konvence"). jestlize $a > b$, klademe $\int_{a}^{b} f(x)\,dx = -\int_{b}^{a} f(x)\,dx$
a navic $\int_{a}^{a} = 0 \forall a \in \mathbf{R}$

Lemma. *Necht $[a,b] \subset \mathbf{R}, D_1, D_2$ jsou dve ruzna deleni $[a,b]$. necht D_1 zjemnuje D_2. Necht f je omezena funkce na $[a,b]$.*

Potom $s(f,D_1) \geq s(f,D_2)$ a $S(f,D_1) \leq S(f,D_2)$

Proof. Necht $D_2 = \{x_0, x_1, \ldots, x_{j-1}, x_j, \ldots, x_n\}$, $D_2 = \{x_0, x_1, \ldots, x_{j-1}, z, x_j, \ldots, x_n\}$MA8
Potom (1) $\inf_{x \in [x_{j-1},x_j]} f(x) \leq \inf_{x \in [x_{j-1},z]}$
(2) $\inf_{x \in [x_{j-1},x_j]} f(x) \leq \inf_{x \in [z,x_j]}$
Tedy $\inf_{x \in [x_{j-1},x_j]} f(x)(x_j - x_{j-1}) \leq \inf_{x \in [x_{j-1},z]}(z - x_{j-1}) + \inf_{x \in [x_j,z]}(x_j - z)$
$\Rightarrow s(f,D_2) \leq s(f,D_1)$
(pro D_1, D_2, ktere se lisi o vic bodu indukci)
Analogicky S. □

LEMMA. *Necht D_1, D_2 jsou libovolna deleni $[a,b] \subset \mathbf{R}$. Necht f je omezena funkce na $[a,b]$. Potom $s(f, D_1) \le S(f, D_2)$.*

PROOF. Necht D je spolecne zjemneni D_1 i D_2. Potom (dle predchoziho Lemmatu)

$s(f, D_1) \le s(f, D) \overset{\text{z definice}}{\le} S(f, D) \le S(f, D_2)$ □

DEFINITION 6.1.3. Necht D je deleni $[a,b]$. $D = \{x_j\}_{j=0}^n$. Pak hodnotu $\nu(D) = \max_{j=1,\ldots,n}(x_j - x_{j-1})$ nazveme **normou deleni**.

THEOREM. *1. (aproximace horniho (dolniho) Riemannova integralu pomoci hornich (dolnich) souctu) TV*
Necht $\{D_n\}$ je posloupnost deleni intervalu $[a,b] \subset \mathbf{R}$ takova, ze $\lim_{n\to\infty} \nu(D_n) = 0$.
Necht f je omezena funkce na $[a,b]$.
Potom

$$\underline{\int_a^b} f(x)\,dx = \sup_{n\in\mathbf{N}} s(f, D_n)$$

a

$$\overline{\int_a^b} f(x)\,dx = \inf_{n\in\mathbf{N}} S(f, D_n)$$

PROOF. MA12
/*zvolim deleni z zavislosti na ε. Pro nej najdu (dostatecne malou ν) D_n. Pak odhadnu rozdil tech deleni pomoci spolecneho deleni.
Zvolme $\varepsilon > 0$ a k nemu D takove, ze $s(f, D) > \underline{\int_a^b} f(x)\,dx - \varepsilon$ - $\exists$ plyne z definice (sup)
Oznacime $\mathcal{D}$ mnozinu intervalu, ktere vznikly z deleni D
tedy $D = \{x_j\}_{j=0}^n \mathcal{D} = \{[x_{j-1}, x_j], j = 1, \ldots, n\}$
Necht $K \in \mathbf{R}$ splnuje: $\forall x \in [a,b] : |f(x)| \le K$ (z omezenosti f)
Zvolme $n_0 \in \mathbf{N}$:
(i)$\nu(D_{n_0}) < \min\{\text{delka} I, I \in \mathcal{D}\} \Rightarrow$zadny interval z $\mathcal{D}_{n_0}$ neobsahuje dva delici body D
(ii)$\nu(D_{n_0}) <?$
$P =$"$D_{n_0} \cup D$"
$P \ldots$sestava z delicich bodu deleni D_{n_0}a D.
Plati, ze $s(f, P) \ge s(f, D)$
$\mathcal{P} \ldots$mnozina intervalu vzniklych z P
$0 \le s(f, P) - s(f, D_{n_0}) = \sum_{I\in\mathcal{P}} \inf_I f * \text{delka}(I) - \sum_{I\in\mathcal{D}_{n_0}} \inf_I f * \text{delka}(I)$
/*vsechny intervaly, ktere neobsahuji zadny delici bod z D se zrusi*/
kolem kazdeho deliciho bodu D
$\le |D|\, 2K\nu(D_{n_0}) < \varepsilon$
$\Rightarrow ? = \frac{\varepsilon}{2K|D|}$
Pak mame:
$s(f, D_{n_0}) > s(f, P) - \varepsilon \ge s(f, D) - \varepsilon \ge s(f, D) - \varepsilon > (R)\underline{\int_a^b} f(x)\,dx - 2\varepsilon$ □

EXAMPLE 6.1.4. (i) $f(x) = 1$ na $[0,1]$: $\int_0^1 f(x)\,dx = 1$ inf = sup
(ii) Dirichletova funkce $D(x) = \begin{cases} 1, x \in \mathbf{Q} \\ 0 \end{cases}$ MA9 sup = 1inf = 0
$\underline{\int_0^1} f(x)\,dx = 0, \overline{\int_0^1} f(x)\,dx = 1 \Rightarrow f \notin R([a,b])$ pro zadne $[a,b]$
(iii) $\exists f$ nespojita v ∞ bodech, $f \in R([a,b])$MA10
$f \in R([0,1]), \int_a^b f(x)\,dx = 0$
(iv) Riemannova funkce $\mathcal{R} \in R([a,b])$ - konecny pocet bodu
(v)$\int_0^1 x^2 dx \overset{?}{=} \frac{1}{3}$ zvolim posloupnost deleni s "krokem" $\frac{1}{n}$
$\nu(D_n) = \frac{1}{n}$
Vezmeme $n \in \mathbf{N}$, scitaci index
MA11
$x_J - x_{j-1} = \frac{1}{n}$
$x_j = \frac{j}{n}, j = 0, \ldots, n$
$\inf = (x_{j-1})^2 = \left(\frac{j-1}{n}\right)^2$
$\sup = \left(\frac{j}{n}\right)^2$
$s(f, D_n) = \sum_{j=1}^n \left(\frac{j-1}{n}\right)^2 \frac{1}{n} = \frac{1}{n^3}\sum_{j=1}^n (j-1)^2 = \frac{1}{n^3}\sum_{j=1}^{n-1} j^2 = \frac{1}{n^3}\left(\frac{n(n+1(2n+1))}{6} - n^2\right)$
$\Rightarrow \lim s(f, D_n) = \frac{1}{3}$
analogicky $\lim S(f, D_n) = \frac{1}{3}$
$\Rightarrow \int_0^1 x^2 dx = \frac{1}{3}$

Dusledek. kdyz $\{D_n\}$ jako V1 a navic
$\lim_{n\to\infty} S(f, D_n) = \lim_{n\to\infty} s(f, D_n) = A \in \mathbf{R}$
Potom $\int = A$.

THEOREM. *2. (kriterium ∃ Riemannova integralu (BC podminka)) LV*
Necht f je ***omezena*** *funkce na intervalu $[a, b]$.*
Pak $f \in R([a, b]) \Leftrightarrow \forall\varepsilon > 0 \exists D$ intervalu $[a, b] : S(f, D) - s(f, D) < \varepsilon$

PROOF. "⇒"
$\varepsilon > 0$ je dano.
$f \in R([a, b]) \Rightarrow \exists$ deleni D_1, D_2 intervalu $[a, b]$:
$S(f, D_1) < (R) \int_{\underline{a}}^{b} f(x)\,dx + \frac{\varepsilon}{2}$
$s(f, D_2) > (R) \int_{\underline{a}}^{b} f(x)\,dx - \frac{\varepsilon}{2}$
Zvolme deleni D tak, ze zjemnuje D_1i D_2
potom $s(f, D_2) \leq s(f, D) \& S(f, D) \leq S(f, D_1)$
$\Rightarrow 0 \leq S(f, 0) - s(f, D) < \varepsilon$
"⇐"
Zvolme $\varepsilon > 0$ a k nemu deleni D splnujici
$S(f, D) - s(f, D) < \varepsilon$ pak plati
$0 \leq \int_a^{\overline{b}} f(x)\,dx - \int_{\underline{a}}^{b} f(x)\,dx \leq S(f, D) - s(f, D) < \varepsilon \Rightarrow \int_a^{\overline{b}} f(x)\,dx = \int_{\underline{a}}^{b} f(x)\,dx \Rightarrow f \in R([a, b])$ □

DEFINITION 6.1.5. Funkce f je **stejnomerne spojita** na intervalu I, jestlize plati
$\forall\varepsilon > 0 \exists\delta > 0 \forall x \in I \forall y \in I : |x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$

NOTE 6.1.6. Je-li f na I stejnomerne spojita $\Rightarrow f$ je na I spojita

EXAMPLE 6.1.7. $f(x) = \frac{1}{x}$je spojita, ale ne stejnomerne

THEOREM. *3˙ (vztah spojitosti a stejnomerne spojitosti) LV*
Necht funkce f je spojita na omezenem ***uzavrenem*** *intervalu $[a, b]$. Pak je f stejnomerne spojita na $[a, b]$.*

PROOF. Sporem.
Necht tedy f je spojita na $[a, b]$ a pritom $\exists\varepsilon > 0 \forall\delta > 0 \exists x, y \in [a, b] : |x - y| < \delta \& |f(x) - f(y)| \geq \varepsilon$
Potom $\forall n \in \mathbf{N} \exists x_n, y_n \in [a, b] : |x_n - y_n| < \frac{1}{n} \& |f(x_n) - f(y_n)| \geq \varepsilon$
MA13
Podle Bolzano-Weistrassovy vety ∃ vybrana posloupnost $\{x_{n_k}\}_{k=1}^{\infty}$ z $\{x_n\}_{n=1}^{\infty}$ splnujici $\lim_{k\to\infty} x_{n_k} = x^* \in \mathbf{R}$
Potom $x^* \in [a, b]$
Dale $\lim_{k\to\infty} y_{n_k} = x^*$ protoze $|y_{n_k} - x_{n_k}| \leq |y_{n_k} - x_{n_k}| + |x_{n_k} - x^*| \leq \underbrace{\frac{1}{n_k}}_{\to 0} + \underbrace{|x_{n_k} - x^*|}_{\to 0}$& policajti

MA14
Funkce f je spojita v bode x^* (vzhledem k $[a, b]$ - muze byt koncovy bod). Potom $\exists\delta > 0$ takove, ze $\forall z \in \mathcal{P}(x^*, \delta) \cap [a, b]$: $|f(z) - f(x^*)| < \frac{\varepsilon}{4}$.
Nyni nalezneme k_0 takove, ze $x_{n_{k_0}}, y_{n_{k_0}} \in \mathcal{P}(x^*, \delta)$
Pak $\varepsilon \leq \left|f\left(x_{n_{k_0}}\right) - f\left(y_{n_{k_0}}\right)\right| \leq \left|f\left(x_{n_{k_0}}\right) - f(x^*)\right| + \left|f(x^*) - f\left(y_{n_{k_0}}\right)\right| \leq \frac{\varepsilon}{2} \Rightarrow$ SPOR □

THEOREM. *4. (riemannovska integrovatelnost spojitych funkci) LV*
Necht funkce f je spojita na ***uzavrenem*** *intervalu $[a, b]$. Pak je $f \in R([a, b])$.*

PROOF. /*omezenost, vypocet*/
Funkce f je spojita na $[a, b] \Rightarrow f$ je omezena.
Uzijeme V2. Zvolime $\varepsilon > 0$. Podle V3 $\exists\delta > 0$ takove, ze $\forall x, y \in [a, b] : |x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$
Zvolme deleni $D = \{x_j\}_{j=0}^{n}$ intervalu $[a, b]$ takove, ze $\nu(D) < \delta$
Pocitejme

$$S(f, D) - s(f, D) = \sum_{j=1}^{n} \left(\sup_{[x_{j-1}, x_j]} f - \inf_{[x_{j-1}, x_j]} f \right) (x_j - x_{j-1}) \leq \sum_{j=1}^{n} \varepsilon (x_j - x_{j-1}) \leq \varepsilon (b - a) \Rightarrow f \in R([a, b])$$

□

THEOREM. *5. (riemannovska integrovatelnost monotonnich funkci) LV*
Necht f je omezena monotonni funkce na $[a, b]$. Pak $f \in R([a, b])$.

PROOF. /*posloupnost deleni*/
BUNO f je neklesajici.
Pouzijeme V2. Zvolme $\varepsilon > 0$.
Necht $D = \{x_j\}_{j=0}^{n}$, kde $x_j = a + j\frac{b-a}{n}, n \in \mathbf{N}, n >? = \frac{(f(b)-f(a))(b-a)}{\varepsilon}$
Pocitejme $S(f, D) - s(f, D) = \sum_{j=1}^{n} \left(\sup_{[x_{j-1}, x_j]} f - \inf_{[x_{j-1}, x_j]} f\right)(x_j - x_{j-1}) \leq \sum_{j=1}^{n} (f(x_j) - f(x_{j-1})) \frac{b-a}{n}$
$= \frac{b-a}{n} \sum_{j=1}^{n} (f(x_j) - f(x_{j-1})) = \frac{b-a}{n} (f(b) - f(a)) < \varepsilon \Rightarrow ? = \frac{(f(b)-f(a))(b-a)}{\varepsilon}$ □

THEOREM. *6. (vlastnosti Riemannova integralu) LV*
(a) Necht $f,g \in R([a,b])$ a $\alpha \in \mathbf{R}$.
Potom $f+g \in R([a,b]), \alpha f \in R([a,b])$ a plati

$$(R)\int_a^b (f+g) = (R)\int_a^b (f) + (R)\int_a^b (g)$$

$$(R)\int_a^b (\alpha f) = \alpha (R)\int_a^b (f)$$

(b) Necht $f,g \in R([a,b])$ a $f \leq g$. Potom

$$(R)\int_a^b (f) \leq (R)\int_a^b (g)$$

(c) aditivita vzhledem k intervalum.
Necht $(a < b < c) \in \mathbf{R}$. Pak plati
(i) $f \in R([a,c]) \Leftrightarrow f \in R([a,b]) \& f \in R([b,c])$
(ii) je-li $f \in R([a,c])$, pak $(R)\int_a^c (f)\,dx = (R)\int_a^b (f)\,dx + (R)\int_b^c (g)\,dx$ ma-li aspon 1 strana smysl

PROOF. (a)
D deleni $[a,b]$. Pak plati
$S(f+g,D) \leq S(f,d) + S(f,g)$, $s(f+g,D) \geq s(f,d) + s(f,g)$
protoze $\sup\{f(x)+g(x), x \in I\} \leq \sup\{f(x), x \in I\} + \sup\{g(x), x \in I\}$
$[\forall x \in I : f(x) \leq \sup_I f, g(x) \leq \sup_I g]$
$\Rightarrow \forall x \in I : f(x) + g(x) \leq \sup_I f + \sup_I g \Rightarrow \sup\{f(x)+g(x), x \in I\} \leq \sup\{f(x), x \in I\} + \sup\{g(x), x \in I\}$
inf analogicky.
$...S(f+g,D) = \sum \sup(f+g) \leq \sum \sup(f) + \sum \sup(g) = S(f,d) + S(f,g)$
Uzijeme V2. Zvolme $\varepsilon > 0$.
Pak $\exists D_1, D_2$ deleni $[a,b]$ takova, ze $S(f,D_1) - s(f,D_1) < \frac{\varepsilon}{2} \& S(g,D_2) - s(g,D_2) < \frac{\varepsilon}{2}$ (protoze $f,g \in \mathbf{R}$)
Necht D je zjemneni D_1 i D_2. Potom $S(f,D) - s(f,D) < \frac{\varepsilon}{2} \& S(g,D) - s(g,D) < \frac{\varepsilon}{2}$
a dale $S(f+g,D) - s(f+g,D) \leq S(f,D) + S(g,D) - s(f,D) - s(g,D) \leq \varepsilon \Rightarrow f+g \in R([a,b])$
Dale plati, ze $s(f,D) + s(g,D) \leq s(f+g,D) + (R)\int_a^b (f+g) \leq S(f+g,D) \leq S(f,D) + S(g,D)$.
Vhodnou volbou D plyne z predchozich nerovnosti prvni rovnost (a)
(b)
$\alpha \geq 0$ $S(\alpha f,D) = \alpha S(f,D)$
$\alpha < 0$ $S(\alpha f,D) = \alpha s(f,D)$
Uzijeme V2. Zvolme $\varepsilon > 0$. K nemu zvolime deleni $D : S(f,D) - s(f,D) < \varepsilon$ Potom plati:
$S(\alpha f,D) - s(\alpha f,D) = |\alpha|(S(f,D) - s(f,D)) \leq |\alpha|\varepsilon \Rightarrow \alpha f \in R([a,b])$
Rovnost zase dokazeme vhodnou volbou D.
(b) vypliva z $S(f,D) \leq S(g,D)$, D je deleni $[a,b]$.
(c)
(i)
"$\Rightarrow$"
V2: $f \in R([a,c]) \Leftrightarrow \forall \varepsilon > 0 \exists D : S(f,D) - s(f,D) < \varepsilon$
Zvolim $\varepsilon > 0, \exists D$, necht $f \in R([a,c])$
Oznacim $D^l = \{\text{body z } D, \text{ktere lezi v } [a,b]\}$
BUNO $b \in D$. Pak $D = \{x_0, \ldots, x_k, \ldots x_n\}$ a $D^l = \{x_0, \ldots, x_k\}$
$S(f,\Delta) - s(f,\Delta) = \sum_{j=1}^k \left(\sup_{[x_{j-1},x_j]} f - \inf_{[x_{j-1},x_j]} f\right)(x_j - x_{j-1}) \leq$
$\sum_{j=1}^n \left(\sup_{[x_{j-1},x_j]} f - \inf_{[x_{j-1},x_j]} f\right)(x_j - x_{j-1}) = S(f,D) - s(f,D) < \varepsilon$
$\overset{V2}{\Rightarrow} f \in R([a,b])$ (analogicky $f \in R([b,c])$)
"$\Leftarrow$" Vezmu posloupnost deleni $\{D_n\}$ intervalu $[a,c]$ tak, aby $b \in D_n \forall n$
$\nu(D_n) \to 0, n \to \infty$
potom $D_n = D_n^l \cup D_n^r$, kde $D_n^l = D_n \cap [a,b]$, $D_n^r = D_n \cap [b,c]$
$S(f,D_n) = S(f,D_n^l) + S(f,D_n^r)$
$s(f,D_n) = s(f,D_n^l) + s(f,D_n^r)$

$$\left\{\begin{array}{c} \lim S(f,D_n) = \lim S(f,D_n^l) + \lim S(f,D_n^r) \\ \lim s(f,D_n) = \lim s(f,D_n^l) + \lim s(f,D_n^r) \end{array}\right\} \overset{Dusledek V1}{\Rightarrow} (R)\int_a^c f(x)\,dx = (R)\int_a^b f(x)\,dx + (R)\int_b^c f(x)\,dx$$

□

THEOREM. *(riemannuv integral a absolutni hodnota) LV*
Necht $f \in R([a,b])$. Pak $|f| \in R([a,b])$ a plati:

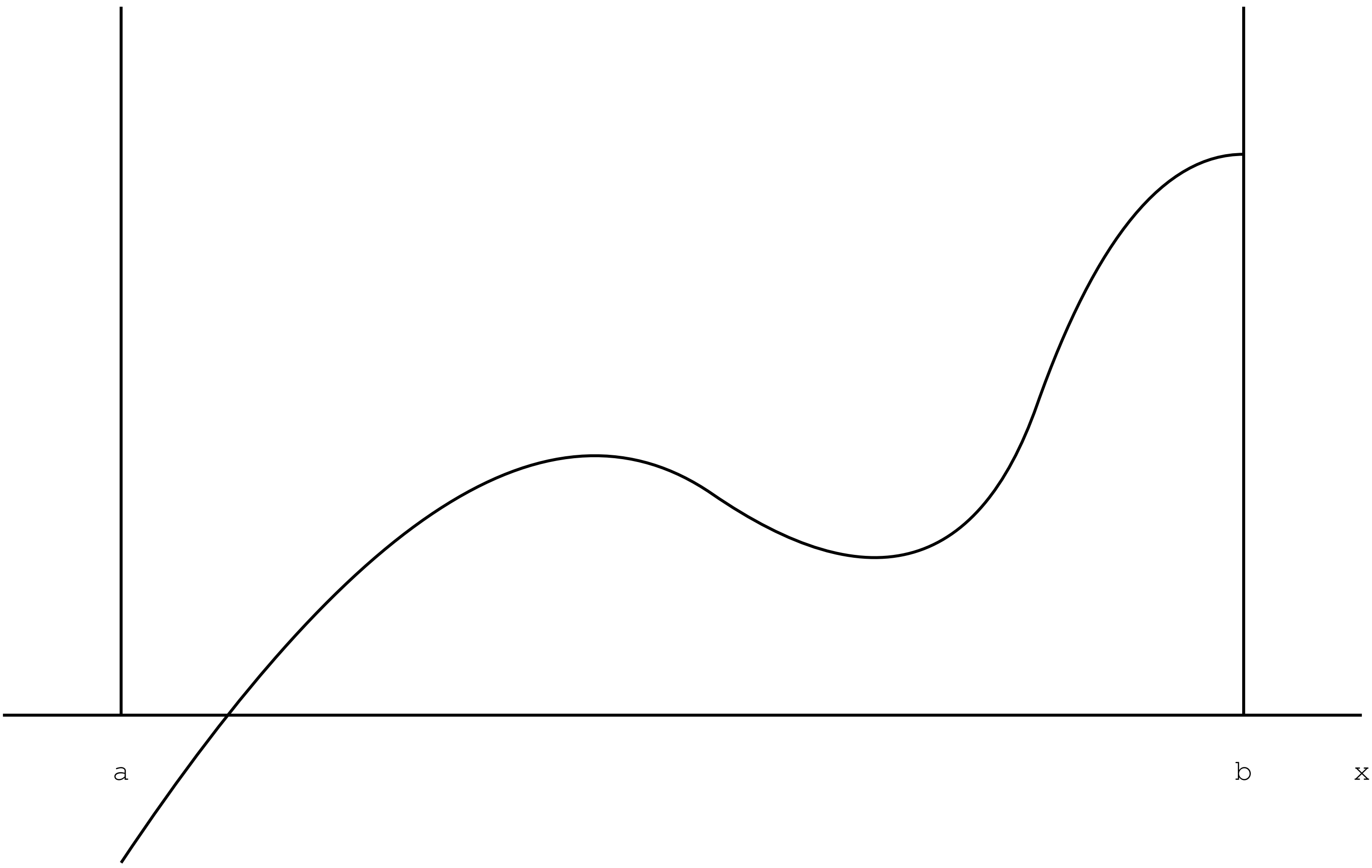
$$\left|\int_a^b f(x)\,dx\right| \leq \int_a^b |f(x)|\,dx$$

PROOF. $f \in R([a,b])$, zvol ε, pak $\exists D : S(f,D) - s(f,D) < \varepsilon$
$S(|f|,D) - s(|f|,D) \leq |S(f,D) - s(f,D)| = S(f,D) - s(f,D) < \varepsilon \Rightarrow |f| \in R([a,b])$
dale vime, ze $f(x) \leq |f(x)|, -f(x) \leq |f(x)| \overset{V6b}{\Rightarrow} \int f(x)\,dx \leq \int |f(x)|\,dx \& -\int f(x)\,dx \leq \int |f(x)|\,dx$
$\Rightarrow \left|\int_a^b f(x)\,dx\right| \leq \int_a^b |f(x)|\,dx$ □

NOTE 6.1.8. obracene: napriklad Dirichletova funkce

6.1.3. Newtonuv integral.

THEOREM. *7. (derivovani riemannova integralu podle horni meze) LV*
(velmi dulezita - spojitost mezi N a R)
Necht I je interval a f je funkce definovana na I splnujici $\forall \alpha, \beta \in I : f \in R([\alpha,\beta])$. *Necht c je libovolny bod z I.*
Pak definujme na I funkci $F(x) = (R)\int_c^x f(t)\,dt \forall x \in I$ *(Karlinska konvence plati i pro* $x < c$*)*
Potom plati:
(i) F je spojita na I.
(ii) je-li f spojita v x_0*, pak* $F\prime(x_0) = f(x_0)$.



PROOF. (i)
Zvolime $y_0 \in I, y_0$ neni pravym krajnim bodem I
Chceme: $\lim_{y \to y_0+}(F(y) - F(y_0)) = 0$
$F(y) - F(y_0) = \int_c^y f(x)\,dx - \int_c^{y_0} f(x)\,dx \overset{V6c}{=} \int_{y_0}^y f(x)\,dx$
vime:
$\exists \delta > 0 : f \in R([y_0, y_0+\delta]) \Rightarrow f$ je omezena na $[y_0, y_0+\delta]$, tedy $\exists m, M \forall y \in [y_0, y_0+\delta] : m < f(y) < M$
Monotonie: $m\left(\underbrace{y - y_0}_{\to 0}\right) \leq \int_{y_0}^y f(x)\,dx \leq M\left(\underbrace{y - y_0}_{\to 0}\right) \forall y \in [y_0, y_0+\delta]$
Policajti pro $y \to y_0+ : \lim_{y \to y_0} \int_{y_0}^y f(x)\,dx = 0$
spojitost zleva analogicky
(ii)
$F\prime(x_0) = \lim_{h \to 0} \frac{F(x_0+h) - F(x_0)}{h} = \lim_{h \to 0} \frac{1}{h}\int_{x_0}^{x_0+h} f(t)\,dt$ pokud limita $\exists$
Nas zajima, zda $F\prime(x_0) - f(x_0) = 0$. Tedy zkoumejme vyraz
$\left(\frac{1}{h}\int_{x_0}^{x_0+h} f(t)\,dt\right) - f(x_0) = \frac{1}{h}\int_{x_0}^{x_0+h}(f(t) - f(x_0))\,dt$
Vime, ze f je spojita v x_0. Tedy ke zvolenemu $\varepsilon \exists \delta : |t - x_0| < \delta \Rightarrow |f(t) - f(x_0)| < \varepsilon$

Odtud plyne, ze pro $h < \delta$: $\left|\frac{1}{h}\int_{x_0}^{x_0+h}(f(t) - f(x_0))\,dt\right| \leq \frac{1}{h}\int_{x_0}^{x_0+h}(|f(t) - f(x_0)|)\,dt < \varepsilon$ //posledni nerovnost - je to sice zrejmy, ale matematicky?

$\Rightarrow |F\prime(x_0) - f(x_0)| < \varepsilon \forall \varepsilon \Rightarrow F\prime(x_0) = f(x_0)$ □

NOTE 6.1.9. (i) f nemusi byt omezena na I ($\frac{1}{x}$na $(0,\ldots)$)
(ii) $\forall \alpha, \beta \in I$
(iii) Tvrzeni V7 (ii) neplati bez predpokladu spojitosti (sign)

dusledek 1 (dluh -V2 z kapitoly 5).

THEOREM. *(existence riemannova integralu pro spojitou funkci) LV*
f spojita na $(a,b) \Rightarrow f$ ma primitivni funkci na (a,b)

PROOF. okamzite plyne z $V7$ f spojita v $\forall x_0 \in (a,b) \Rightarrow F\prime(x_0) = f(x_0)$ □

dusledek 2. (dobre pro vypocty)
Necht f spojita na $[a,b]$, $F = \int f$
Pak $(R)\int_a^b = \lim_{x\to a-} F(x) - \lim_{x\to a+} F(x)$

PROOF. f spojita na $[a,b]$
BUNO f spojita na $(a-1, b+1)$
Def $G(x) = G(a) + \int_a^x f(t)\,dt, x \in (a-1, b+1)$
/*$= G(a) + [F(t)]_a^x = G(a) + F(x) - F(a)$*/
Potom $G\prime = f$ na $(a-1, b+1)$ - plyne z V7 a $(G(a))\prime = 0$.
Tim spise $G\prime = f$ na (a,b).
Protoze $F\prime = f \& G\prime = f$ na (a,b), nutne plati $F(x) = G(x) + d, d \in \mathbf{R}, x \in (a,b)$
$\lim_{x\to a+} F(x) = \lim_{x\to a+} G(x) + d = G(a) + d$
$\lim_{x\to b-} F(x) = \lim_{x\to b-} G(x) + d = G(b) + d$
$\lim_{x\to b-} F(x) - \lim_{x\to a+} F(x) = G(b) - G(a)$
$G(b) = G(a) + \int_a^b f(t)\,dt$
$\Rightarrow \lim_{x\to b-} F(x) - \lim_{x\to a+} F(x) = (R)\int_a^b f(t)\,dt$ □

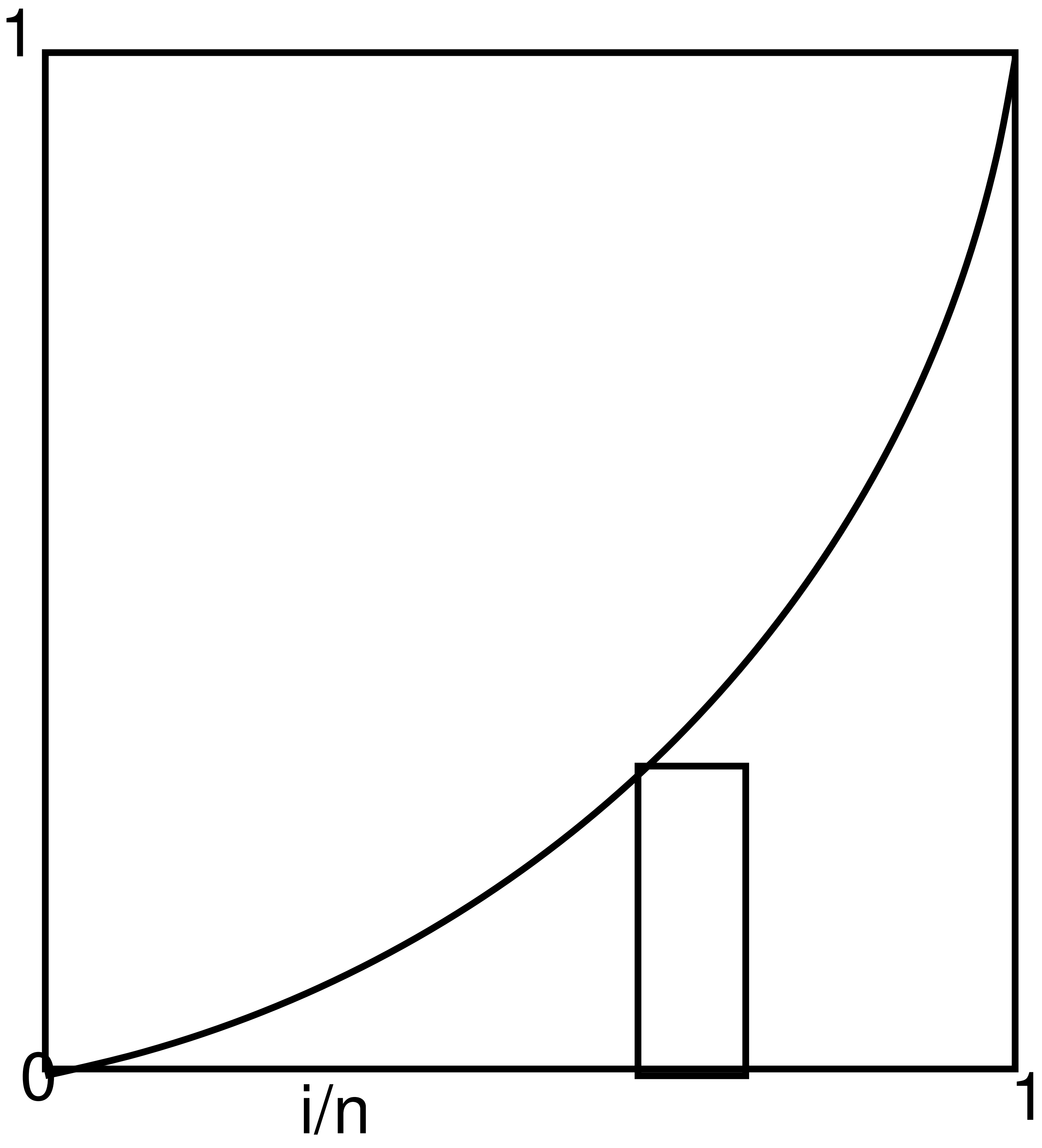
EXAMPLE 6.1.10. $(R)\int_0^1 x^2 dx = \left[\frac{x^3}{3}\right]_{x=0}^{x=1} = \frac{1}{3}$ $F(x) = \frac{x^3}{3}$

DEFINITION 6.1.11. **(klicovy pojem)**
Rekneme, ze funkce f ma na $[a,b]$ **Newtonuv integral** ($f \in N([a,b])$), jestlize f ma na (a,b) primitivni funkci F a $\exists \lim_{x\to a+} F(x) \in \mathbf{R}, \lim_{x\to b-} F(x) \in \mathbf{R}$. Potom definujeme

$$(N)\int_a^b f(x)\,dx = \lim_{x\to a+} F(x) - \lim_{x\to b-} F(x)$$

NOTE 6.1.12. (i) Nekdy se definuje $N((a,b))$, otevreny interval, napr. neomezeny
(ii) Nekdy se misto vlastnich limit pozaduje pouze aby rozdil limit mel smysl
(iii) $R([a,b])$ ... riemannovsky int. fce
$N([a,b])$ ... newtonovsky int. fce
$C([a,b])$ ... spojite fce
Platí: $f \in C([a,b]) \Rightarrow f \in R([a,b]) \& f \in N([a,b]) \& (N)\int_a^b f(x)\,dx = (R)\int_a^b f(x)\,dx$ (Plyne z V4 a V7)

EXAMPLE 6.1.13. (i) $f(x) = \frac{1}{\sqrt{x}}, x \in (0,1], f(0) = 0$
na $(0,1)$ ma f prim. fci a to $F(x) = 2\sqrt{x}$
tudiz $(N)\int_0^1 \frac{dx}{\sqrt{x}} = [2\sqrt{x}]_0^1 = 2$
ale $f \notin R([a,b])$ (neni omezena)
(ii) $f(x) = sign(x), x \in [-1,1]$
$(R)\int_{-1}^1 f(x) = 0$
ale $f \notin N([-1,1])$ (f neni darbouxovska, a tedy nema na $(-1,1)$ primitivni funkci)
(iii) existuje priklad (obtizny) funkce f takove, ze f je omezena, ale presto $f \in N([a,b]) \setminus R([a,b])$
(iv)(Stolzovou vetou pro $p \in \mathbf{N}$) $\lim_{n\to\infty} \frac{1^p + \ldots + n^p}{n^{p+1}} = \lim_{n\to\infty} \frac{1}{n}\left(\left(\frac{1}{n}\right)^p + \left(\frac{2}{n}\right)^p + \ldots + \left(\frac{n}{n}\right)^p\right)$
$= \lim_{n\to\infty} S(x^p D_n) =$



Example 6.1.14. x^p spoj. na $[0,1] \Rightarrow \exists (R) \int_0^1 x^p dx = (N) \int_0^1 x^p dx$
$= (R) \int_0^1 x^p dx = (N) \int_0^1 x^p dx = \left[\frac{x^{p+1}}{p+1} \right]_0^1 = \frac{1}{p+1}, p > -1$

Theorem. *8. (per partes pro urcity integral) LV*
Necht $f, g, f\prime, g\prime$ jsou spojite funkce na $[a, b]$
Potom
$\int_a^b f(x) g\prime(x) dx = [f(x) g(x)]_a^b - \int_a^b f\prime(x) g(x) dx$
kde $[f(x)]_a^b = \lim_{x \to b-} f(x) g(x) - \lim_{x \to a+} f(x) g(x)$ $(= f(b) g(b) - f(a) g(a))$

Proof. Vsechno je spojite $\Rightarrow$ vsechny integraly $\exists$.
Tvrzeni plyne z obdobnych vet pro primitivni funkce (kapitola 5) □

Theorem. *9. (substituce pro urcity integral) LV*
(i) Necht f je spojita na $[a, b]$, necht $\varphi : [\alpha, \beta] \to [a, b]$, necht φ ma na $[\alpha, \beta]$ spojitou prvni derivaci.
Potom plati $\int_\alpha^\beta f(\varphi(t)) \varphi\prime(t) dt = \int_{\varphi(\alpha)}^{\varphi(\beta)} f(x) dx$

*(ii) Necht f je spojita na $[a,b]$, necht $\varphi : [\alpha,\beta] \to [a,b]$ je **na**, necht φ ma na $[\alpha,\beta]$ spojitou nenulovou prvni derivaci. Potom $\int_a^b f(x)\,dx = \int_{\varphi^{-1}(a)}^{\varphi^{-1}(b)} f(\varphi(t))\,\varphi\prime(t)\,dt$*

Proof. (i) f spoj $\Rightarrow$ ma na (a,b) prim. funkci F.
Podle V o substituci pro primitivni funkci: $\int f(\varphi(t))\,\varphi\prime(t)\,dt = F(\varphi(t)) + c$
To jest $(F \bullet \varphi)\prime = (f \bullet \varphi)\,\varphi\prime$ Tudiz
$\int_\alpha^\beta f(\varphi(t))\,\varphi\prime(t)\,dt = [F \bullet \varphi]_\alpha^\beta = [F]_{\varphi(\alpha)}^{\varphi(\beta)} = \int_{\varphi(\alpha)}^{\varphi(\beta)} f(x)\,dx$
(ii) $(f \bullet \varphi)\,\varphi\prime$ je spojita $\Rightarrow$ ma prim. funkci G
Podle V o substituci pro prim. funkce: $\int f(x)\,dx = G(\varphi^{-1})(x) + c$, tedy $\left(G \bullet \varphi^{-1}\right)\prime = f$
Tudiz $\int_a^b f(x)\,dx = \left[G \bullet \varphi^{-1}\right]_a^b = [G]_{\varphi^{-1}(a)}^{\varphi^{-1}(b)} = \int_{\varphi^{-1}(a)}^{\varphi^{-1}(b)} f(\varphi(t))\,\varphi\prime(t)\,dt$ □

Example 6.1.15. (jsou mozne i jine zpusoby)
(i)schema: $y = \varphi(x)$
$dy = \varphi\prime(x)\,dx$

x	a	b			
y	$\varphi(a)$	$\varphi(b)$			

$\Rightarrow \int_{\varphi(a)}^{\varphi(b)}$

Example 6.1.16. $\int_0^{\frac{\pi}{2}} \cos^3 x \sin x dx = I$
$y = \cos x$
$dy = -\sin x dx$
$I = \int_1^0 y^3(-dy) = \int_0^1 y^3 dy = \frac{1}{4}$
(ii) (jednodussi) schema:
$x = \varphi(t)$
$dx = \varphi\prime(t)\,dt$

x	a	b			
t	$\varphi^{-1}(a)$	$\varphi^{-1}(b)$			

Example 6.1.17. plocha $\frac{1}{2}$kruhu
$I = \int_{-r}^{r} \sqrt{r^2 - x^2} dx$
$x = r \sin t$
$dx = r\cos(t)\,dt$
$I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{r^2 - r^2\sin^2 t} r \cos t dt = r^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2 t dt = r^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1+\cos 2t}{2} dt =$
$(\int \frac{1+\cos 2t}{2} = \frac{\sin 2t}{4} + c)$
$= r^2\left(\frac{\pi}{2} + \left[\frac{\sin 2t}{4}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\right) = r^2\left(\frac{\pi}{2} + \frac{\sin\pi - \sin(-\pi)}{4}\right) = \frac{\pi r^2}{2}$

6.2. aplikace urciteho integralu

6.2.1. Plochy rovinnych objektu. MA17

Example 6.2.1. Vypoctete plochu sevrenou krivkami $y = 2^x, y = 2, x = 0$
$= \int_0^1 (2 - 2^x)\,dx = \ldots = 2 - \frac{1}{\log 2}$

6.2.2. Limity posloupnosti.

Example 6.2.2. $\lim_{n\to\infty} \sum_{i=1}^n \frac{n}{n^2+i^2} = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^n \frac{1}{1+\left(\frac{i}{n}\right)^2} = \lim_{n\to\infty} s(f,D), f(x) = \frac{1}{1+x^2}$

$\overset{\text{spojitost}}{=} (R)\int_0^1 \frac{1}{1+x^2} dx = (N)\int_0^1 \frac{1}{1+x^2} dx = \frac{\pi}{4}$

6.2.3. Limity funkci.

Example 6.2.3. $\lim_{x\to 0+} \frac{\int_0^x \cos(t^2)\,dt}{x} \overset{l\prime H \frac{0}{0}}{=}$ (spojitost primitivni funkce, karlinska konvence)

$\overset{V7}{=} \lim_{x\to 0+} \frac{\cos(x^2)\,dt}{1} \overset{VOLslFCE}{=} 1$

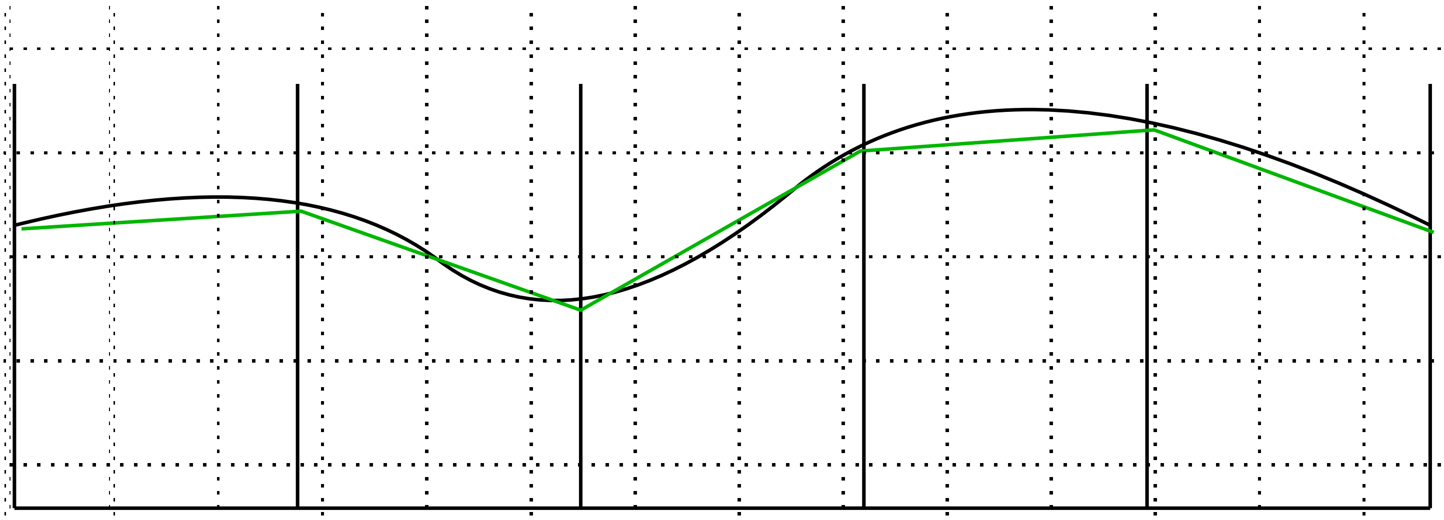
6.2.4. Delka krivky.

Definition 6.2.4. Necht f je spojita na $[a,b]$. Potom **delkou krivky** dane funkci f nazvu vyraz $L(f) = \sup_D \sum_{j=1}^n \sqrt{(f(x_j) - f(x$
/*nevypada to jako deleni - krok*/
$\sum_{j=1}^n \sqrt{(f(x_j) - f(x_{j-1}))^2 + (x_j - x_{j-1})^2} = \sum_{j=1}^n (x_j - x_{j-1}) \sqrt{1 + \left(\frac{f(x_j) - f(x_{j-1})}{x_j - x_{j-1}}\right)^2}$
domnenka: delka krivky $= \int_a^b \sqrt{1 + (f\prime(x))^2} dx$

THEOREM. *10. (delka krivky) LV*
Necht f ma na $[a,b]$ spojitou prvni derivaci, pak plati
delka krivky $= L(f) = \int_a^b \sqrt{1+(f\prime(x))^2}dx$

PROOF. Oznacme $g(x) = \sqrt{1+(f\prime(x))^2}$

Zvolme D deleni $[a,b]$ a oznacme $L(f,D) = \sum_{j=1}^n (x_j - x_{j-1})\sqrt{1+\left(\frac{f(x_j)-f(x_{j-1})}{x_j-x_{j-1}}\right)^2}$ □



PROOF. Podle Lagrangeovy vety $\exists \xi_j \in (x_{j-1}, x_j) : L(f,D) = \sum_{j=1}^n (x_j - x_{j-1})\sqrt{1+(f\prime(\xi_j))^2}$
$s(g,D) \leq L(f,D)$ - protoze $L(f,D)$ ma misto inf ξ
$\leq S(g,D)$
podle definice: $L(f) = \sup_D L(f,D)$
podle definice: $\int_a^b = \sup_D s(g,D)$
$\Rightarrow \int_a^b g(x)\,dx \leq \sup_D L(f,D) = L(f)$
jinymi slovy $\int_a^b \sqrt{1+(f\prime(x))^2}dx \leq L(f)$
Prespokladejme pro spor: $L(f) > \int_a^b g(x)\,dx$
Zvolim posl. deleni $\{D_n\}$ tak, aby:
D_1 zjemnovalo D, D_{n+1} zjemnovalo D_n a $\lim_{n\to\infty} \nu(D_n) = 0$
Potom: $L(f,D) \leq L(f,D_1) \leq \ldots \leq$
$\lim_{n\to\infty} s(g,D_n) = \int_a^b g(x)\,dx$ (g je spojita)
$s(g,D_n) \leq L(f,D_n)$
$\Rightarrow \lim s(g,D_n) \leq \lim L(f,D_n) \leq \lim S(g,D_n) = \int_a^b g(x)\,dx$ - spor □

THEOREM. *11. (zobecneni V10 pro krivky v $\mathbf{R}^n$) (bez dukazu)*
Necht $\varphi : [a,b] \Rightarrow \mathbf{R}^n$, tzn. $\varphi(t) = (\varphi_1(t), \varphi_2(t), \ldots, \varphi_n(t))$
Necht $\varphi\prime_1(t), \varphi\prime_2(t), \ldots, \varphi\prime_n(t)$ jsou spojite na $[a,b]$
Potom $L(\varphi) = \int_a^b \sqrt{\varphi\prime_1^2(t) + \varphi\prime_2^2(t) + \ldots + \varphi\prime_n^2(t)}dt$

NOTE 6.2.5. Je-li nahodou φ popsatelna grafem funkce, pak mame:
$\varphi(t) = [t, f(t)]$, pak $L(\varphi) = \int_a^b \sqrt{1+f\prime^2(t)}dt$ (V10)

EXAMPLE 6.2.6. $\varphi(t) = (r\cos t, r\sin t)$ (kruznice v 0 s polomerem r)
$t \in [0, 2\pi]$
$L(f) = \sqrt{\varphi_1\prime(t)^2 + \varphi_2\prime(t)^2} = \int_0^{2\pi} \sqrt{r^2\cos^2 t + r^2\sin^2 t}dt = r\int_0^{\pi} dt = 2\pi r$
Pak $L(f)$

NOTE 6.2.7. $\exists$ Peannova krivka, ktera zobrazuje $[0,1]$ na $[0,1]^2$ a ktera je na $[0,1]$ spojita
Neni prosta a nema spojitou derivaci. Proto je ve vete omezujici predpoklad

6.2.5. Objem a povrch rotacniho telesa.

THEOREM. *12. (objem a povrch rotacniho telesa) (bez dukazu)*
Necht f je na $[a,b]$ nezaporna a spojita, oznacme
$T = T(x,y,z) = \left\{[x,y,z,], \sqrt{y^2+z^2} \leq f(x)\right\}$
Potom objem $V(T) = \pi\int_a^b (f(x))^2\,dx$
a
povrch plaste $S(T) = 2\pi\int_a^b (f(x))\sqrt{a+(f\prime(x))^2}dx$ *(je nutne zohlednit sikmost)*

NOTE 6.2.8. dukaz: objem = limita spalku k 0
povrch: skosena obruc

6.2.6. konvergence rad.

THEOREM. *13. (integralni kriterium konvergence rad)*
Necht f je nezaporna, spojita a nerostouci na $[n_0-1,\infty)$ pro nejake $n_0\in\mathbf{N}$
Potom rada $\sum_{n=n_0}^{\infty} f(n)$ konverguje $\Leftrightarrow \exists$ (konecny) Newtonuv integral $(N)\int_{n_0}^{\infty} f(x)\,dx<\infty$

PROOF. /*hezky pouziti predpokladu*/
Zvolim $N>n_0, D=\{n_0,n_0+1,\dots,N\}$ (deleni intervalu $[n_0,N]$)
Potom $s(f,D)=\sum_{j=n_0+1}^{N} f(j)$
a $S(f,D)=\sum_{j=n_0}^{N-1} f(j)$
f je spojita na $[n_0,N]\Rightarrow \exists(R)\int_{n_0}^{N}, \exists(N)\int_{n_0}^{N}$, rovnaji se
$\Rightarrow s(f,D)=\sum_{j=n_0+1}^{N} f(j)\le (R)\int_{n_0}^{N} f(x)\,dx=(N)\int_{n_0}^{N} f(x)\,dx\le \sum_{j=n_0}^{N-1} f(j)=S(f,D)$
$\Rightarrow \lim_{N\to\infty}\sum_{j=n_0+1}^{N} f(j)\le \lim_{N\to\infty}\int_{n_0}^{N} f(x)\,dx\le \lim_{N\to\infty}\sum_{j=n_0}^{N-1} f(j)$
$s<\infty \Leftrightarrow (N)\int_{n_0}^{\infty} f(x)\,dx<\infty$ □

EXAMPLE 6.2.9. Pro ktera $\alpha\in\mathbf{R}$ konverguje rada $\sum_{n=2}^{\infty}\frac{1}{n(\log n)^{\alpha}}$?
Pro $\alpha\le 0$ diverguje
pro $\alpha>0$ int. kriterium $\Rightarrow$ konverguje $\Leftrightarrow \int_2^{\infty}\frac{dx}{x(\log x)^{\alpha}}=\int_{\log 2}^{\infty}\frac{dy}{y^{\alpha}}<\infty \Leftrightarrow \alpha>1$

6.2.7. stirlingův vzorec. $\lim_{n\to\infty}\frac{n!}{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n}=1$ - cela prednaska, mozna se stihne (ne, nestihne)

6.2.8. dukaz V o zavedeni exponencialy.

THEOREM. *(o existenci logaritmu) LV*
(V13 z Kapitoly 4) (zavedeni exponencialy)
Existuje prave jedna realna funkce "exp" definovana na $\mathbf{R}$ splnujici $\forall x,y\in\mathbf{R}$:
(i) $\exp(x+y)=(\exp x)(\exp y)$
(ii) $\exp x\ge 1+x$

PROOF. (oklikou pres logaritmus)
Def $F(x)=\int_1^x\frac{dt}{t}, x\in(0,\infty)$
Potom V7 $\Rightarrow$ F je spojita na $(0,\infty)$, F je rostouci (V7 $F\prime=\frac{1}{x}>0$)
(*)$\lim_{x\to\infty}F(x)=\infty$ (napr proto, ze $\int_1^x\frac{dt}{t}\ge\sum_{n=2}^{N}\frac{1}{n}$(diverguje))MA19
$\lim_{N\to\infty}\int_1^N\frac{dt}{t}=\infty$
(*)$F(xy)=F(x)+F(y)$
$F(xy)=\int_1^{xy}\frac{dt}{t}=\int_1^x\frac{dt}{t}+\int_1^{xy}\frac{dt}{t}=$
(x je pevne cislo) substituce: $s=\frac{t}{x}ds=\frac{dt}{x}$
$=\int_1^x\frac{dt}{t}+\int_1^y\frac{ds}{s}=F(x)+F(y)$
(*)$F(x)\le x-1$ pro $x\in(0,\infty)$
$x>1\Rightarrow\frac{1}{t}<1\forall t\in(1,x)\Rightarrow\int_1^x\frac{dt}{t}\overset{V6ii}{\le}\int_1^x dt=x-1$
$x<1\Rightarrow\frac{1}{t}>1\forall t\in(x,1)\Rightarrow\int_x^1\frac{dt}{t}>\int_x^1 dt=1-x\ /(-1)$
$\Rightarrow\int_1^x\frac{dt}{t}<x-1$
$F(1)=0=1-1$
(*)$\lim_{x\to 0+}F(x)=-\infty$
$\lim_{x\to 0+}F(x)=\lim_{y\to\infty}F\left(\frac{1}{y}\right)=\lim_{y\to\infty}F(1)-F(y)=-\lim_{y\to\infty}F(y)=-\infty$
- tim jsme zavedli logaritmus - F je def, spojita, rostouci na $(0,\infty)$, $F:(0,\infty)\to\mathbf{R}$ (zobrazeni je **na**)
$\Rightarrow\exists F^{-1}:\mathbf{R}\Rightarrow(0,\infty)$
polozime $\exp x=F^{-1}(x)$.
Potom: expje def, spojita, rostouci na $\mathbf{R}$, $\lim_{x\to-\infty}\exp x=0$, $\lim_{x\to\infty}\exp x=\infty$
$F(xy)=F(x)+F(y)/\exp$
$xy=\exp(F(x)+F(y))\,\forall x,y>0$
subst: $s=F(x), t=F(y)\Rightarrow x=\exp s, y=\exp t$
$\Rightarrow(\exp s)(\exp t)=\exp(s+t)$
$\Rightarrow\exp(x+y)=(\exp x)(\exp y)$
$F(x)\le x-1, x\in(0,\infty)/\exp$ (je rostouci $\Rightarrow\le$se nemeni)
$x\le\exp(x-1)\,\forall x\in(0,\infty)$
$y=x-1\Rightarrow y+1\le\exp y, y\in(-1,\infty)$
pro $y\le -1: y+1\le 0<\exp y$ (nerovnost plati trivialne)
$\Rightarrow$ EXISTENCE exp
Dokazali jsme, ze ma-li funkce exp vlastnosti (i) a (ii), potom
$\exp(x)=\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n \overset{\text{V o jednoznacnosti limity}}{\Rightarrow}$ JEDNOZNACNOST exp □

NOTE 6.2.10. jiny dukaz (primo)
def $\exp x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$
- koverguje absolutne $\forall x$
- $\exp(x+y) = \exp(x)\exp(y)$ (cviceni - Cauchyuv soucin rad + binomicka veta)
- $\exp 0 = 1$
- $\exp(-x) = \frac{1}{\exp(x)}$
$\Rightarrow \exp x > 0 \forall x \in \mathbf{R}$
- $\exp x > 1 + x$ pro $x \geq 0$ zrejme
pro $x \leq -1$ zrejme $(1 + x \leq 0 < \exp x)$
$x \in (-1,0) : y \in (0,1) \Rightarrow \sum_{n=0}^{\infty} \frac{y^n}{n!} \leq \sum_{n=0}^{\infty} y^n \overset{\text{geom.rada}}{=} \frac{1}{1-y} \Rightarrow \exp y \leq \frac{1}{1-y} \forall y \in (0,1)$
$\Rightarrow \exp(-y) \geq 1 - y \Rightarrow \exp x \geq 1 + x \forall x \in (-1,0)$

-

NOTE 6.2.11. Jak zavest $\sin, \cos, \pi$?
$A(x) = \int_0^x \frac{dt}{\sqrt{1-t^2}}, \frac{\pi}{2} = A(1), \sin = A^{-1}$, ale blbe se odvozujou souctovy vzorce
nebo:
def $B(x) = \int_0^x \frac{dt}{1+t^2}, \left(B^{-1}\right)\prime = \frac{1}{\cos^2}$
nebo (se znalosti komplexnich cisel):
$e^{it} = \cos t + i \sin t$
$e^{int} = (\cos t + i \sin t)^{in} = \cos nt + i \sin nt$

CHAPTER 7

Metricke prostory

7.0.9. motivace.

(1) rozsireni pojmu spojitosti do obecnejsiho kontextu
(2) vysetrovani extremu funkci vice promennych
(3) diferencialni pocet funkci vice promennych
(4) vysetrovavysetrovani extremu funkci definovanych na jinych mnozinach nez na **R**
(5) rozsireni pojmu konvergence do obecnejsiho kontextu

7.1. zakladni pojmy

DEFINITION 7.1.1. **(klicovy pojem) Metrickym prostorem** nazveme dvojici (P,ρ), kde P je neprazdna mnozina a ρ je funkce $\rho:(P\times P)\to[0,\infty)$ splnujici $\forall x,y,z\in P$:
(i) $\rho(x,y)=0\Leftrightarrow x=y$
(ii) $\rho(x,y)=\rho(y,x)$ (symetrie)
(iii) $\rho(x,z)\leq\rho(x,y)+\rho(y,z)$
funkci ρ nazyvame **metrikou** na P

EXAMPLE 7.1.2. (i) ($\mathbf{R^n}$, eukleidovska metrika)
$\mathbf{R^n}=\{x=[x_1,\dots x_n]\}$
eukleidovska metrika $\rho_e(x,y)=\sqrt{\sum_{i=1}^n(x_i-y_i)^2}$ (nekdy take l_2-metrika)
overme, ze je to metrika: (i),(ii) zrejme
()(iii): $\sqrt{\sum_{i=1}^n(x_i-y_i)^2}\leq\sqrt{\sum_{i=1}^n(x_i-z_i)^2}+\sqrt{\sum_{i=1}^n(z_i-y_i)^2}$
oznac: $a_i=x_i-z_i, b_i=z_i-y_i$, potom $a_i+b_i=x_i-y_i$
tedy $\sqrt{\sum_{i=1}^n(a_i+b_i)^2}\leq\sqrt{\sum_{i=1}^n(a_i)^2}+\sqrt{\sum_{i=1}^n(b_i)^2}$
Cauchyova nerovnost: $\sum a_ib_i\leq\sqrt{\left(\sum a_i^2\right)\left(\sum b_i^2\right)}$
$\sqrt{\sum_{i=1}^n(a_i+b_i)^2}=\sqrt{\sum_{i=1}^n a_i^2+2\sum_{i=1}^n a_ib_i+\sum_{i=1}^n b_i^2}\leq\sqrt{\sum_{i=1}^n a_i^2+2\sqrt{\left(\sum_{i=1}^n a_i^2\right)\left(\sum_{i=1}^n b_i^2\right)}+\sum_{i=1}^n b_i^2}$
$=\sqrt{\left(\sqrt{\sum_{i=1}^n a_i^2}+\sqrt{\sum_{i=1}^n b_i^2}\right)^2}=\sqrt{\sum_{i=1}^n(a_i)^2}+\sqrt{\sum_{i=1}^n(b_i)^2}$
(ii) ($\mathbf{R^n}$, newyorkska metrika)
nekdy l_1-metrika
$\rho_1(x,y)=\sum_{i=1}^n|x_i-y_i|$
vlastnosti metriky: cviceni
(iii) ($\mathbf{R^n}$, maximova metrika)
nekdy l_∞-metrika
$\rho_\infty(x,y)=\max_{i=1,\dots n}|x_i-y_i|$
(iv) ($\mathbf{R^n}$, ruska metrika)
- jedno vyjimecne mesto a
$\rho(x,y)=\begin{cases}0,x=y\\ \rho(x,a)+\rho(a,y)\end{cases}$
(v)$P=\zeta([a,b])$ (mnozina vsech spojitych funkci na $[a,b]$)
ρ_s ¯supremova metrika: $\rho_s(f,g)=\sup_{x\in[a,b]}|f(x)-g(x)|$
(i)OK
(ii)OK
(iii) $|x-y|\leq|x-z|+|z-y|\,\forall x,y,z\in\mathbf{R}$
Tedy: pro 3 funkce f,g,h :
$\forall x\in[a,b]:|f(x)-g(x)|\leq|f(x)-h(x)|+|h(x)-g(x)|$
$\leq\sup_{y\in[a,b]}|f(y)-h(y)|+\sup_{y\in[a,b]}|h(y)-g(y)|=\rho(f,h)-\rho(h,g)$
$\forall x\in[a,b]:|f(x)-g(x)|\leq\rho(f,h)-\rho(h,g)$ prejdu k supremu na leve strane
$\Rightarrow\sup_{x\in[a,b]}|f(x)-g(x)|\leq\rho(f,h)-\rho(h,g)\Rightarrow\rho(f,g)\leq\rho(f,h)-\rho(h,g)\,\forall f,g,h\in\zeta[a,b]$
(vi) $P=\zeta[a,b]$
ρ_i - integralni metrika
$\rho_i(f,g)=(R)\int_a^b|f(x)-g(x)|\,dx$ - overeni podobne
(vii) P - lib. neprazdna metrika
ρ_d - diskretni metrika: $\rho_d(x,y)=\begin{cases}0:x=y\\1:x\neq y\end{cases}$

DEFINITION 7.1.3. Necht (P,ρ) je metricky prostor. Necht $x \in P$ a necht $r > 0$. **Otevrenou kouli o stredu x a polomeru** r nazveme mnonzinu $B(x,r) = \{y \in P : \rho(x,y) < r\}$.

Uzavrenou kouli o stredu x a polomeru r nazveme mnonzinu $\overline{B}(x,r) = \{y \in P : \rho(x,y) \leq r\}$.

EXAMPLE 7.1.4. (i) $(\mathbf{R}^2, \text{eukl.})$ $B([0,0],1)$ (jednotkova koule) MA20
(ii) $(\mathbf{R}^2, \text{newyork.})$ $B([0,0],1)$ (jednotkova koule) MA22
(iii) $(\mathbf{R}^2, \max.)$ $B([0,0],1)$ (jednotkova koule) MA21
(iv) ruska
(iv) $(P,\rho_d) : B(x,r) = \begin{cases} \{r\}, r \leq 1 \\ P, r > 1 \end{cases}$

DEFINITION 7.1.5. **(klicovy pojem)** Necht (P,ρ) je metricky prostor, $G \subseteq P$. Rekneme, ze G je **otevrena**, jestlize $\forall x \in G \exists r > 0 : B(x,r) \subset G$ - /*s kazdym bodem se do ni vejde i naka kulicka*/
(klicovy pojem) Necht $F \subseteq P$. Rekneme, ze F je **uzavrena**, jestlize $P \setminus F$ je otevrena

NOTE 7.1.6. (i) Otevrena koule $B(x,r)$ je vzdy otevrena mnozina
Necht $y \in B(x,r)$. Def $r_0 = r - \rho(x,y)$. Potom (snad) $B(y,r_0) \subset B(x,r)$
Chceme dokazat: $z \in B(y,r_0) \Rightarrow z \in B(x,r)$
Vime $z \in B(y,r_0)$, tj. $\rho(y,z) < r_0$.
Pak $\rho(z,x) \leq \rho(z,y) + \rho(y,x) < r_0 + \rho(x,y) = r \Rightarrow z \in B(x,r)$.
(ii) Uzavrena koule $\overline{B}(x,r)$ je uzavrena mnozina
Pokud $\overline{B}(x,r) = P : \emptyset$ je otevrena
Jinak necht $y \in P \setminus \overline{B}(x,r)$. Potom $\rho(x,y) > r$.
Zvolime $r_0 = \rho(x,y) - r > 0$.
Potom (snad) $B(y,r_0) \subset P \setminus \overline{B}(x,r)$
$\rho(z,y) < r_0$
Necht $z \in B(y,r_0)$. Pak $\rho(z,x) \geq \rho(x,y) - \rho(y,z) > \rho(x,y) - r_0 = r$

THEOREM. *1. (vlastnosti otevrenych mnozin) LV*
(P,ρ) je metricky prostor.
(i) Potom $\emptyset, P$ jsou otevrene
(ii) $G_1,\ldots,G_n$ je konecne mnoho otevrenych mnozin, pak $\bigcap_{i=1}^n G_i$ je otevrena mnozina
(iii) $\{G_\alpha\}_{\alpha\in I}$ je system otevrenych mnoziny, pak $\bigcup_{\alpha\in I}$ je otevrena mnozina

PROOF. (i) pro $\emptyset$ plati trivialne
pro $\forall x \in P$ staci zvolit libovolne $r > 0$, pak $B(x,r) \subset P$
(ii) $G_1,\ldots,G_n$ jsou otevrene. Oznacime $G = \bigcap_{i=1}^n G_i$. Necht $x \in G$. Potom $x \in G_i \forall i = 1,\ldots,n$ a tedy $\forall i \exists r_i > 0 : B(x,r_i) \subset G_i$.
Zvolime $r = \min_{i=1,\ldots,n} r_i > 0$. Potom $B(x,r) \subset G_i \forall i = 1,\ldots,n \Rightarrow B(x,r) \subset G$ tedy G je otevrena.
(iii) $\{G_\alpha\}_{\alpha\in I}$, def. $G = \bigcup_\alpha G_\alpha$. Zvol $x \in G$. Pak $\exists \alpha : x \in G_\alpha$, G_α je otevrena $\Rightarrow \exists r : B(x,r) \subset G_\alpha \subset G$, tedy G je otevrena. □

THEOREM. *2. (vlastnosti uzavrenych mnozin) LV*
(P,ρ) je metricky prostor.
(i) Potom $\emptyset, P$ jsou uzavrene
(ii) $F_1,\ldots,G_n$ je konecne mnoho uzavrenych mnozin, pak $\bigcup_{i=1}^n F_i$ je uzavrena mnozina
(iii) $\{F_\alpha\}_{\alpha\in I}$ uzavrene mnoziny, pak $\bigcap_{\alpha\in I}$ je uzavrena mnozina

PROOF. (i) $P \setminus \emptyset = P$ je otevrena dle V1
$P \setminus P = \emptyset$ je otevrena dle V1
(ii) $\bigcup_{i=1}^n F_i$ je otevrena, nebot $P \setminus \bigcup_{i=1}^n F_i = \bigcap_{i=1}^n (P \setminus F_i)$ je otevrena dle V1
(iii) $\bigcap_\alpha F_\alpha$ je uzavrena, nebot $P \setminus \bigcap_\alpha F_\alpha = \bigcup_\alpha (P \setminus F_\alpha)$ je otevrena dle V1 □

EXAMPLE 7.1.7. $G_n = \left(0, 1 + \frac{1}{n}\right)$ otevrene mnoziny v $(\mathbf{R}, |x-y|)$
$\bigcap_{n\in\mathbf{N}} G_n = (0,1]$ neni otevrena

DEFINITION 7.1.8. Necht (P,ρ) je metricky prostor, $A \subset P$. Pak definujeme **uzaver** mnoziny A
$\overline{A} = \bigcap \{F, \text{kde } F \text{ je uzavrena} \& F \supset A\}$
Vnitrkem mnoziny A nazyvame mnozinu
$A^\circ = IntA = \bigcup \{G, \text{kde } G \text{ je otevrena} \& G \supset A\}$

EXAMPLE 7.1.9. (i) $A = (0,1]$, pak $\overline{A} = [0,1]$ a $IntA = (0,1)$
(ii) $A = \mathbf{Q}$, pak $\overline{A} = \mathbf{R}$, $IntA = \emptyset$

NOTE 7.1.10. $\overline{A}$ je nejmensi uzavrena mnozina obsahujici A,
$IntA$ je nejvetsi otevrena mnozina, obsazena v A

znaceni. vzdalenost bodu x od mnoziny $A \subset P$
definujeme takto: $\rho(x,A) = \inf\{\rho(x,y), y \in A\}$

NOTE 7.1.11. $x \in A \Rightarrow \rho(x,A) = 0$
tedy $\{y \in P, \rho(y,A) = 0\} \supset A$
$\overset{?}{\subset}$ NE - $(P,\rho) = (\mathbf{R}, |x-y|)$, $A = (0,1)$, $y = 1$ pak $\rho(y,A) = 0 \& y \notin A$

THEOREM. *3. (vlastnosti uzaveru) LV*
Necht (P,ρ) je metricky prostor.
(i) $F \subset P$ je uzavrena $\Rightarrow \overline{F} = F$
(specialne $\overline{\emptyset} = \emptyset$, $\overline{P} = P, \forall A \subset P : \overline{\overline{A}} = \overline{A}$)
(ii) $A \subset B \Rightarrow \overline{A} \subset \overline{B}$
(iii) $\overline{A \cup B} = \overline{A} \cup \overline{B}$
(iv) $\overline{A} = \{y \in P, \rho(y,A) = 0\}$

PROOF. (i) OK
(ii) $A \subset B \Rightarrow A \subset \overline{B}, \overline{B}$ uzaver $\Rightarrow \overline{A} \subset \overline{B}$
(iii) $A \subset A \cup B \Rightarrow A \subset \left(\overline{A} \cup \overline{B}\right)$
$B \subset A \cup B \Rightarrow B \subset \left(\overline{A} \cup \overline{B}\right)$
$\Rightarrow (A \cup B) \subset \left(\overline{A} \cup \overline{B}\right)$
V2ii $\overline{A} \cup \overline{B}$ je uzavrena
$\Rightarrow \overline{(A \cup B)} \subset \overline{A} \cup \overline{B}$
opacna inkluze

$$\left.\begin{array}{l} A \subset (A \cup B) \overset{(ii)}{\Rightarrow} \overline{A} \subset \overline{A \cup B} \\ B \subset (A \cup B) \overset{(ii)}{\Rightarrow} \overline{B} \subset \overline{A \cup B} \end{array}\right. \Rightarrow \overline{A \cup B} = \overline{A} \cup \overline{B}$$

(iv) oznacime $M == \{y \in P, \rho(y,A) = 0\}$, chceme $M = \overline{A}$
k tomu staci dokazat:
*1 $M \supset A$ (OK)
*2 M uzavrena
*3$M \subset \overline{A}$
*1 & *2 $\Rightarrow M \subset \overline{A}$
*2: $y \in (P \setminus M) \Rightarrow \rho(y,A) > 0$, tedy $\exists r > 0 : B(y,r) \cap A = \emptyset$
chceme ale dokazat, ze $\exists r_1 : B(y,r_1) \cap M = \emptyset$
Polozime $r_1 = \frac{r}{2}$, necht $a \in A, z \in B(y,r_1)$
Potom $\rho(a,z) \geq \rho(y,a) - \rho(y,z) \geq r - \frac{r}{2} = \frac{r}{2}$
$\Rightarrow \rho(z,A) \geq \frac{r}{2} \forall z \in B(y,r_1) \Rightarrow B(y,r_1) \subset (P \setminus M) \Rightarrow (P \setminus M)$ otevrena $\Rightarrow M$ uzavrena
*3: Necht ne. Potom $\exists y \in \left(M \setminus \overline{A}\right)$
Pak $\rho(y,A) = 0$ (nebot $y \in M$)
Zaroven ale $y \notin \overline{A}$. Tedy $y \in \left(P \setminus \overline{A}\right)$, coz je ot. mnozina
Tede $\exists r > 0 : B(y,r) \subset \left(P \setminus \overline{A}\right)$
Pak ale $\rho(y,A) > 0$ - spor □

7.2. konvergence v metrickych prostorech

DEFINITION 7.2.1. Necht (P,ρ) je metricky prostor. Necht $\{x_n\}_{n=1}^{\infty}$ je posloupnost prvku z P. Necht $y \in P$. Rekneme, ze $\{x_n\}$ **konverguje** k y (v metrice ρ), jestlize $\lim_{n\to\infty} \rho(x_n,y) = 0$

znaceni. $x_n \to y$ (pripadne $x_n \overset{\rho}{\to} y$)

nebo $\lim x_n = y$

EXAMPLE 7.2.2. konvergence v diskretni metrice - pouze konstantni

THEOREM. *4. (vlastnosti konvergence) LV*
(P,ρ) metricky prostor.
(i) Necht $\{x_n\}_{n=1}^{\infty}$ splnuje: $\exists n_0 \forall n \geq n_0 : x_n = y$ pro nejake $y \in P$. Potom $x_n \to y$
(ii) Limita posloupnosti je urcena jednoznacne
(iii) Necht $x_n \to y$ a $\{x_{n_k}\}_{k=1}^{\infty}$ je vybrana z $\{x_n\}_{n=1}^{\infty}$
Potom $x_{n_k} \to y$

PROOF. (i) Pro $n \geq n_0 : \rho(x_n,y) = \rho(y,y) = 0 \Rightarrow \lim_{n\to\infty} \rho(x_n,y) \to 0$
(ii) spor: Necht $x_n \to y_1 \& x_n \to y_2$ Pak

$$\rho(y_1,y_2) \leq \underbrace{\rho(y_1,x_n)}_{\to 0} + \underbrace{\rho(y_2,x_n)}_{\to 0} \Rightarrow \rho(y_1,y_2) = 0 \overset{\text{(vlastnost metriky)}}{\Rightarrow} y_1 = y_2$$

(iii) $\rho(x_{n_k},y)$ je vybrana posloupnost (kladnych cisel) z posloupnosti $\rho(x_n,y)$
Tedy $\lim_{k\to\infty} \rho(x_{n_k},y) = \lim_{n\to\infty} \rho(x_n,y)$ □

EXAMPLE 7.2.3. (i) v $(\mathbf{R}, |x-y|)$ splyva konvergence s konvergenci definovanou v zimnim semestru

(ii) v $(\mathbf{R}^k, \rho_2)$ plati: $x_n \to y \Leftrightarrow \forall i = 1, \ldots, k : x_n^i \to y^i$
dukaz: "$\Rightarrow$" $\sqrt{\sum_{i=1}^k (x_n^i - y^i)^2} \geq |x_n^i - y^i| \, \forall i = 1, \ldots, k$
"$\Leftarrow$" aritmetika limit
(iii) ekvivalence konvergence v "normalnich" metrikach
$\sum_{i=1}^k |x_n^i - y^i| \leq \sqrt{\left(\sum_{i=1}^k 1\right)\left(\sum_{i=1}^k |x_n^i - y^i|^2\right)} = \sqrt{k}\sqrt{\sum_{i=1}^k (x_n^i - y^i)^2} \leq \sqrt{k}\sqrt{k} \max |x_n^i - y^i|$
$= k \max_{i=1,\ldots,k} |x_n^i - y^i| \leq k \sum_{i=1}^k |x_n^i - y^i|$
$\Rightarrow$ v $\mathbf{R}^n$ plati: $x_n \overset{\rho_1}{\to} y \Leftrightarrow x_n \overset{\rho_2}{\to} y \Leftrightarrow x_n \overset{\rho_\infty}{\to} y$
$x_n^i = (x_n^1, \ldots, x_n^k)$

THEOREM. *5. (charakterizace uzavrenych mnozin pomoci konvergence)*
Necht (p, ρ) je metricky prostor, $F \subset P$.
Potom F je uzavrena $\Leftrightarrow \forall \{x_n\}_{n=1}^\infty \subset F, x_n \to y, y \in P \Rightarrow y \in F$
"kam si dokonverguju, to je moje"

PROOF. "$\Rightarrow$"
F uzavrena, $\{x_n\} \subset F, x_n \to y \in P$. Chceme: $y \in F$
Vime: $\rho(x_n, y) \to 0$. Tedy $\rho(y, F) = 0$ Nebot $\rho(y, F) \leq \rho(y, x_n) \to 0$.
$\overset{V3(iv)}{\Rightarrow} y \in \overline{F} \overset{F \text{ uzavrena}}{\Rightarrow} y \in F = \overline{F}$
"$\Leftarrow$" neprimo: chceme dokazat: necht F neni uzavrena, potom $\exists x_n \subset F, x_n \to y \& y \notin F$
F neni uzavrena $\Rightarrow F \neq \overline{F} \Rightarrow \exists y \in (\overline{F} \setminus F)$
Vime (V3(iv)) $\rho(y, F) = 0$ (nebot $y \in \overline{F}$)
$\rho(y, F) = \inf_{x \in F} \rho(y, x) = 0$. Takze (z vlastnosti infima):
$\forall n \in \mathbf{N} \exists x_n \in F : \rho(y, x_n) < \frac{1}{n}$
Tim ziskavame posloupnost $\{x_n\} \subset F$, ktera splnuje $\rho(x_n, y) \to 0, y \notin F$ □

7.2.1. "spojitost" v metrickem prostoru. f spojita na uzavrenem intervalu $[a, b] \subset \mathbf{R}$, pak je to OK-funkce
?? na metrickem prostoru
samotna uzavrenost :-($\mathbf{R}$ je uzavrena
$(0, 1)$
v prostoru (P, ρ), $P = (0, 1)$ je.
v prostoru $(\mathbf{R}, |x - y|)$, $P = (0, 1)$ neni.
Vsechny mnoziny v diskretnim jsou otevrene ($\Rightarrow$uzavrene)

DEFINITION 7.2.4. **(klicovy pojem)**
Necht (P, ρ) je metricky prostor, $K \subset P$.
Rekneme, ze mnozina K je **kompaktni**, jestlize z kazde posloupnosti $\{x_n\} \subset K$ lze vybrat konvergentni podposloupnost $\{x_{n_k}\}$, jejiz limita je prvkem K.

EXAMPLE 7.2.5. (i) $((0, 1), |x - y|)$ neni kompaktni - napr. z $\left\{\frac{1}{n}\right\}$ nelze vybrat konvergentni
(ii) konecna mnozina je vzdycky kompaktni - aspon jeden prvek se v posloupnosti opakuje ∞krat
(iii) $([a, b], |x - y|)$ je kompaktni
(nebo $K = [a, b]$ je kompaktni v prostoru $(\mathbf{R}, |x - y|)$)
(Bolzano-Weierstrass: lze vybrat konvergentni)
$\{x_n\} \subset [a, b] \Rightarrow \exists \{x_{n_k}\} \to x \in \mathbf{R} \overset{K \text{ uzavrena}}{\Rightarrow} x \in [a, b] \Rightarrow K$ kompaktni
(iv) Dusledek $K \subset \mathbf{R}$ je uzavrena a omezena (v $(\mathbf{R}, |x - y|)$), pak je kompaktni

THEOREM. *6. (vlastnosti kompaktnich mnozin) LV*
Necht (P, ρ) je metricky prostor. Necht $K \subset P$ je kompaktni. Potom
(i) K je uzavrena
(ii) K je ***omezena*** *(tj. $\exists x_0 \in P \exists r : K \subset B(x_0, r)$)*
(iii) jestlize $F \subset K$ je uzavrena, pak F je kompaktni

PROOF. (i) - z definic
Chceme dokazat: $\{x_n\} \subset K, x_n \to x \in P \Rightarrow x \in K$
Necht $\{x_n\} \subset K, x_n \to x \in P$
K kompaktni $\Rightarrow \exists \{x_{n_k}\} \to y \in K$
$V4(iii) \Rightarrow x = y \Rightarrow x \in K$
(ii) sporem
Zvolime $x_0 \in P$
neni omezena $\Rightarrow \forall n \in \mathbf{N} \exists x_n \in K, \rho(x_0, x_n) > n$
K kompaktni $\Rightarrow \exists \{x_{n_k}\} \to y \in K$, tedy $\rho(x_{n_k}, y) \to 0$
tedy: pro kazde $k \in \mathbf{N}$
$\underbrace{n_k}_{\to\infty} < \rho(x_0, x_{n_k}) \leq \underbrace{\rho(x_0, y)}_{const} + \underbrace{\rho(y, x_{n_k})}_{\to 0}$ SPOR

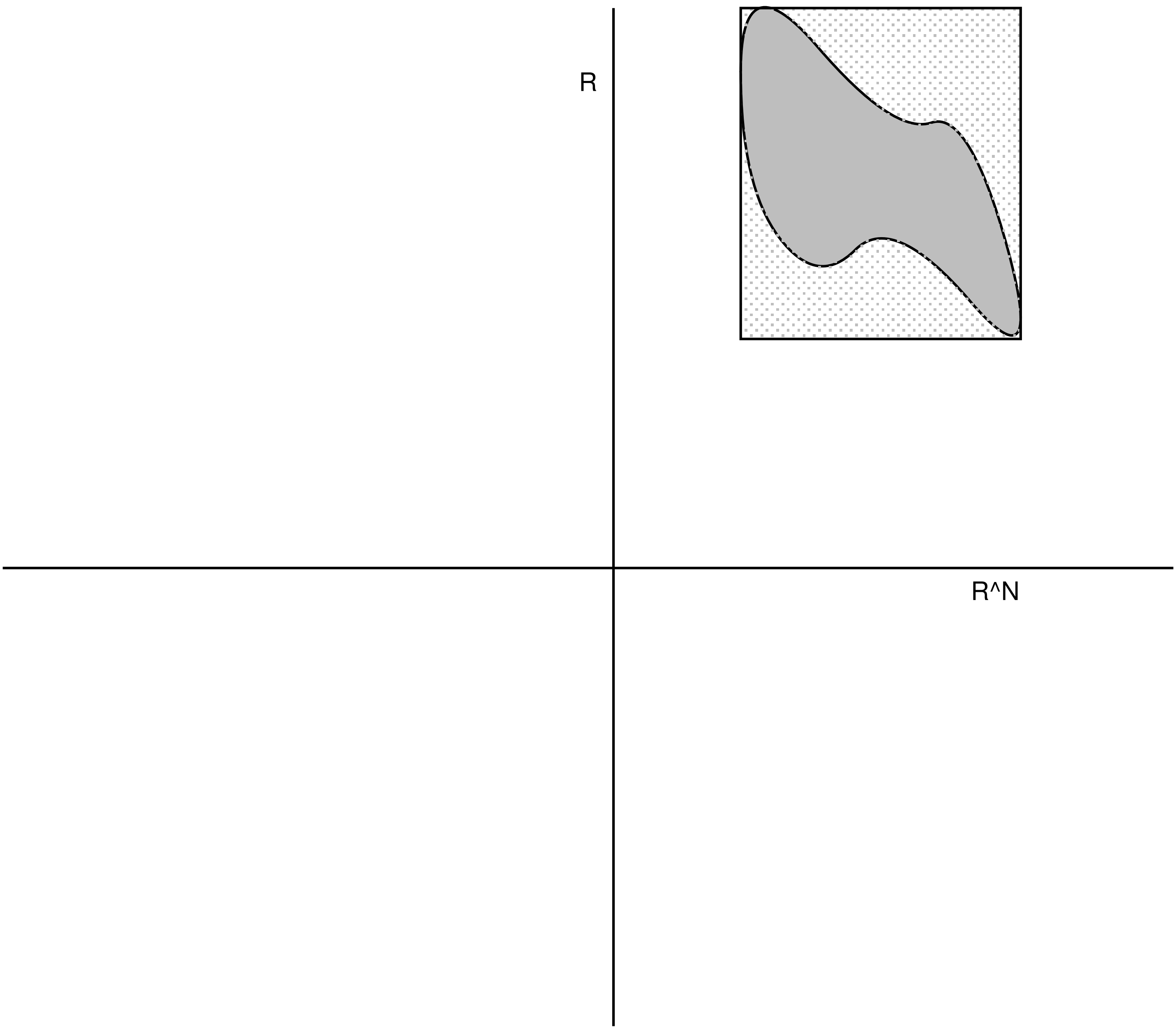
(iii)
$F \subset K$, uzavrena
Necht $\{x_n\} \subset F$
Pak $\{x_n\} \subset K$. K je kompaktni $\Rightarrow \exists \{x_{n_k}\} \to y \in K$
F uzavrena $\Rightarrow y \in F$ □

CVICENI - jak vypadaji komp. mnoziny v diskretnim prostoru.

Example 7.2.6. Uzavrena jednotkova koule $\overline{B}(0,1)$ v $(\zeta[0,1], \rho_{\sup})$
$\overline{B}(0,1) = \left\{ g \in \zeta[0,1], \sup_{x\in[0,1]} |g(x)| \leq 1 \right\}$
Def. $g_n(x) = \begin{cases} 2^n x \text{ pro } x \in \left[0, \frac{1}{2^n}\right] \\ 1 \text{ pro } x \in \left[\frac{1}{2^n}, 1\right] \end{cases}$
Pro $n > m \in \mathbf{N}$
$\rho(g_n, g_m) = \sup_{x\in[0,1]} |g_n(x) - g_m(x)| \geq \left|g_n\left(\frac{1}{2^n}\right) - g_m\left(\frac{1}{2^n}\right)\right| = \left|1 - \frac{1}{2^{n-m}}\right| \geq \frac{1}{2}$
Potom nelze vybrat $\{g_{n_k}\}$ tak, aby $g_{n_k} \to h$, nebot jinak by $\exists k_0 \in \mathbf{N} : \forall j, k \geq k_0 : \rho(g_{m_k}, h) < \frac{1}{4} \& \rho\left(g_{m_j}, h\right) < \frac{1}{4}$, ale pritom
$\frac{1}{2} \leq \rho\left(g_{n_j}, g_{n_k}\right) \leq \rho\left(g_{n_j}, h\right) + \rho(h, g_{n_k}) < \frac{1}{4} + \frac{1}{4}$

Theorem. *7. (charakterizace kompaktnich mnozin v $\mathbf{R}^N$) TV*
Mnozina $K \subset \mathbf{R}^N, N \in \mathbf{N}$ (s eukleidovskou metrikou) je kompaktni $\Leftrightarrow$ K je omezena a uzavrena.

Proof. Staci dokazat “$\Leftarrow$” ($\Rightarrow$ uz mame)
/*dovela lehky - staci opakovat “po slozkach”
indukci, krok: mnozinu osekat uzavrenejma intervalama na n-krychli, ta predpokladem
a pak vybrat kvg, podpodposloupnost pro $n+1$-ni slozku (ktera je taky kompaktni)*/
- indukci podle N.
Pro $N = 1$ uz mame B-W priklad (iii)
Krok $n \to n+1$ □



Proof. $K \subset \mathbf{R}^{N+1}$ omezena a uzavrena
konecna $\Rightarrow \exists$uzavrene intervaly $I_1, \ldots, I_{N+1}$ tak, ze $K \subset (I_1 \times \ldots \times I_N \times I_{N+1})$
konvergence v $\mathbf{R}^N$ - konvergence po slozkach $x_n = [a_n, b_n] \in \mathbf{R}^N \times \mathbf{R}, a_n \in \mathbf{R}^N, b_n \in \mathbf{R}$
Oznacme $A = I_1 \times \ldots \times I_N$ a $B = I_{N+1}$
Pak $[a_n, b_n] \in A \times B$
Mnozina A je zrejme omezena a je take uzavrena - nebot je to soucin uzavrenych intervalu a konvergence v $\mathbf{R}^N$ je po slozkach. Podle indukcniho predpokladu je A kompaktni.
Tedy z posloupnosti $\{a_n\} \subset A$ lze vybrat $a_{n_k} \to a \in A$
Z $x_n = [a_n, b_n]$ jsme vybrali $x_{n_k} = [a_{n_k}, b_{n_k}]$ a vime, ze $a_{n_k} \to a \in A$
$b_{n_k} \in \mathbf{R}, b_{n_k} \in B$, B je kompaktni v $\mathbf{R}$. Tedy mohu znovu vybrat z posloupnosti $\{b_{n_k}\}$ vybrat $\left\{b_{n_{k_j}}\right\}$ takovou, ze $b_{n_{k_j}} \to b \in B$
Tedy $x_{n_{k_j}} = \left[a_{n_{k_j}}, b_{n_{k_j}}\right] \to [a, b]$ (konvergence po slozkach)
K je uzavrena $\Rightarrow x = [a, b] \in K \Rightarrow$ K je kompaktni. □

7.3. spojita zobrazeni na metrickych prostorech

$(P, \rho), (Q, \sigma)$ dva metricke prostory
f funkce $f : (P, \rho) \to (Q, \sigma)$

Definition 7.3.1. Necht $(P, \rho), (Q, \sigma)$ jsou metricke prostory, $f : (P, \rho) \to (Q, \sigma), M \subset \mathcal{D}(f), x_0 \in M$.
(klicovy pojem) Rekneme, ze f je **spojita v** x_0 **vzhledem k** M jestlize $\forall \varepsilon > 0 \exists \delta > 0 \forall x \in B_\rho(x_0, \delta) \cap M : f(x) \in B_\sigma(f(x_0), \varepsilon)$
Rekneme, ze f je **spojita na** M **vzhledem k** M jestlize je spojita v kazdem bode $x_0 \in M$ vzhledem k M

NOTE 7.3.2. (i) pro $M = [a, b]$ definice splyva s definici ze zimy MA
(ii) Pozor na pasti: $f = sign(x)$ je spojita na mnozine $M = (-1, 0) \cup (0, 1)$ vzhledem k P

7.3.1. analogie okoli.
v **R**: okoli ... limita funkce v bode ... spojitost na intervalu
(jednostranne verze)
v metrickem prostoru: spojitost na mnozine a v bode ...okoli ... $B(x, \varepsilon)$
jednostranne ... nema smysl - obecne nemusi byt usporadany
prstencove okoli: $B(x, \varepsilon) \setminus \{x\}$

DEFINITION 7.3.3. (P, ρ) metricky prostor, $x_0 \in P$. Rekneme, ze x_0 je **izolovany bod** P, jestlize $\exists \varepsilon > 0$ tak, ze $B(x_0, \varepsilon) \cap P = \{x_0\}$.

EXAMPLE 7.3.4. (i) $P = (1, 2) \cup \{3\}$ pak bod $\{3\}$ je izolovany (s metrikou $|x - y|$)
(ii) $(P, \rho_d) \Rightarrow$ vsechny body izolovane
Necht $(P, \rho), (Q, \sigma)$ jsou metricke prostory, $f : P \to Q, M \subset P, x_0 \in M$. Necht x_0 neni izolovany bod M.
Rekneme, ze f **ma v bode** x_0 **vzhledem k** M **limitu** $y_0 \in Q$ (znacime $\lim_{x \to x_0, x \in M} f(x) = y_0$), jestlize
$\forall \varepsilon > 0 \exists \delta > 0 \forall x \in (B_\rho(x_0, \delta) \setminus \{x_0\}) : f(x) \in B_\sigma(y_0, \varepsilon)$

NOTE 7.3.5. Funkce je automaticky spojita v izolovanych bodech.
Tedy napr $f\,(P, \text{diskr.}) \to (Q, \sigma)$ je vzdy spojite vzhledem k P

THEOREM. *8. (charakterizace spojitych zobrazeni) LV*
Necht (P, ρ) a (Q, σ) jsou 2 metricke prostory. Necht $f : P \to Q$, $f(P) = Q$
Potom jsou nasledujici 3 vyroky ekvivalentni:
(i) f je spojita na P vzhledem k P
(ii) pro kazdou otevrenou mnozinu $G \subset Q$ je mnozina $f^{-1}(G)$ otevrena v P
(iii) pro kazdou uzavrenou mnozinu $F \subset Q$ je mnozina $f^{-1}(G)$ uzavrena v P

PROOF. (i)$\Rightarrow$(ii) /*1 a 2 pomerne lehky, ale chce si zkusit*/
Neht f je spojita na P vzhledem k P. Necht $G \subset Q$ je otevrena
Chceme $f^{-1}(G)$ je otevrena. Necht $x_0 \in f^{-1}(G)$
Pak $f(x_0) \in G$, G je ot. $\Rightarrow \exists \varepsilon : B_\sigma(f(x_0), \varepsilon) \subset G$
spojitost: k tomuto $\varepsilon \exists \delta : f(B_\rho(x_0, \delta)) \subset B_\sigma(f(x_0), \varepsilon) \subset G$ /aplikujeme f^{-1}
$B_\rho(x_0, \delta) \subset f^{-1}(B_\sigma(f(x_0), \varepsilon)) \subset f^{-1}(G)$
Tedy k x_0 jsme nasli $\delta : B_\rho(x_0, \delta) \subset f^{-1}(G)$, tedy $f^{-1}(G)$ je otevrena.
(ii)$\Rightarrow$(i)
Zvolim $\varepsilon > 0, x_0 \in P$
Budeme zkoumat $B_\sigma(f(x_0), \varepsilon)$
To je otevrena mnozina v Q.
Vezmeme $f^{-1}(B_\sigma(f(x_0), \varepsilon))$
Dle (ii) je i tato mnozina otevrena (v P)
$x_0 \in f^{-1}(B_\sigma(f(x_0), \varepsilon))$ (trivialne nebot $f(x_0) \in B_\sigma(f(x_0), \varepsilon)$)
Tedy $\exists \delta : B_\rho(x_0, \delta) \subset f^{-1}(B_\sigma(f(x_0), \varepsilon))$ /aplikuj f
$f(B_\rho(x_0, \delta)) \subset (B_\sigma(f(x_0), \varepsilon))$ Tedy f je spojita v x_0.
(ii)$\Rightarrow$(iii)
Necht $F \subset Q$ je uzavrena. Potom $Q \setminus F$ je otevrena. Tedy $f^{-1}(Q \setminus F)$ je otevrena v P

$$f^{-1}(F) = f^{-1}(Q \setminus (Q \setminus F)) \overset{(cviceni)}{=} f^{-1}(Q) \setminus f^{-1}(Q \setminus F) = P \setminus f^{-1}(Q \setminus F),$$

coz je mnozina uzavrena.
(iii)$\Rightarrow$(ii) analogicky □

NOTE 7.3.6. Spojitym **obrazem** otevrene mnoziny nemusi byt mnozina otevrena.
(priklad $\sin x$ nebo $f : (P, diskr.) \to (Q, \sigma)$
spoj. obraz **uzavreneho intervalu** je **uzavreny interval**. (protoze uzavreny interval je kompaktni)
Problem: Je spojitym obrazem kompaktu kompakt?

THEOREM. *9. (Heine) LV*
Necht (P, ρ) a (Q, σ) jsou 2 metricke prostory. Necht $f : P \to Q, M \subset P, x_0 \in M$
Potom jsou nasledujici 2 vyroky ekvivalentni:
(i) f je spojita v x_0 vzhledem k M
(ii) Pro kazdou posloupnost $\{x_n\} \subset M$ takovou, ze $x_n \overset{\rho}{\to} x_0$ plati

$$f(x_n) \overset{\sigma}{\to} f(x_0)$$

jiny zapis

$$f \text{ je spojita v } x_0 \text{ vzhledem k } M \Leftrightarrow \left[x_n \overset{\rho}{\to} x_0\right] \Rightarrow f(x_n) \overset{\sigma}{\to} f(x_0)$$

PROOF. (i)⇒(ii)

Necht f je spojita v x_0 vzhledem k M. a necht $\{x_n\} \subset M$ a $x_n \overset{\rho}{\to} x_0$.

Zvolime $\varepsilon > 0$, k nemu $\exists\delta > 0$ takove, ze
$f(B_\rho(x_0,\delta) \cap M) \subset B_\sigma(f(x_0),\varepsilon)$ (ze spojitosti)
K tomuto $\delta > 0$ nutne $\exists n_0 \in \mathbf{N}$ tak, ze $\forall n \geq n_0 : \rho(x_n, x_0) < \delta$ tedy $x_n \in B_\rho(x_0,\delta)\,\forall n \geq n_0$
Tudiz $f(x_n) \in B_\sigma(f(x_0),\varepsilon)\,\forall n \geq n_0$
Dokazali jsme, ze
$\forall\varepsilon\exists n_0 : \forall n \geq n_0 : f(x_n) \in B_\sigma(f(x_0),\varepsilon)$
to jest $f(x_n) \to f(x_0)$.
(ii)⇒(i) sporem
Necht plati (ii) a pritom f neni spojita
$\exists\varepsilon\forall\delta\exists x \in B_\rho(x_0,\delta) \cap M$ tak, ze $f(x) \notin B_\sigma(f(x_0),\varepsilon)$
Specialne (pro $\delta = \frac{1}{n}$) mame:
$\exists\varepsilon\forall n \in \mathbf{N}\exists x_n \in B_\rho\left(x_0, \frac{1}{n}\right) \cap M$ takove, ze $f(x_n) \notin B_\sigma(f(x_0),\varepsilon)$
Tvrdime: $x_n \to x_0$ (plyne z $\rho(x_0,x_n) < \frac{1}{n} \to 0$)

Tedy dle (ii) $f(x_n) \overset{\sigma}{\to} f(x_0)$, tedy $\sigma(f(x_n), f(x_0)) \to 0$ $(n \to \infty)$

Zaroven ma platit: $\sigma(f(x_n), f(x_0)) > \varepsilon\forall n \in \mathbf{N}$ - SPOR □

THEOREM. *10. (spojity obraz kompaktni mnoziny) LV*
Necht (P,ρ) a (Q,σ) jsou 2 metricke prostory. Necht $f : (P,\rho) \to (Q,\sigma)$ spojite vzhledem k P.
Necht $K \subset P$ je kompaktni. Pak $f(K)$ je kompaktni

PROOF. Chceme: $\{y_n\} \subset f(K)$, pak $\exists$ vybrana $y_{n_k} \to y \in f(K)$
Vime: $y_n = f(x_n)$ pro nejake $x_n \in K \forall n \in \mathbf{N}$
K je kompaktni, tedy $\exists$ vybrana podposloupnost $x_{n_k} \to x \in K$
Podle V9: $f(x_{n_k}) \to f(x) \in Q$
Polozime-li $y_{n_k} = f(x_{n_k})$ a $y = f(x)$, mame $y_{n_k} \to y$.
Protoze $x \in K$, $y \in f(K)$
Tedy K je kompakt. □

THEOREM. *11. (nabyvani extremu na kompaktu) LV*
Necht (P,ρ) je metricky prostor a necht f je funkce definovana na P s hodnotami v $\mathbf{R}$.
Necht $K \subset P$ je kompaktni, necht f je na K spojita vzhledem ke K. (na $\mathbf{R}$ uvazujeme metriku $|x-y|$)
Potom f nabyva na K sveho minima i sveho maxima.

PROOF. $f(k)$ je kompaktni mnozina v $\mathbf{R} \Rightarrow f(k)$ je neprazdna, uzavrena a omezena v $\mathbf{R}$
$\Rightarrow \exists s = \inf f(k), t = \sup f(k), -\infty < s \leq t < \infty$
Z vlastnosi infima: $\exists y_n \in f(k) : y_n \to s$ Tedy $s \in f(k)$ nebot K je uzavrena.
Analogicky pro t. □

THEOREM. *12. (spojitost a omezenost) LV*
Necht (P,ρ) je metricky prostor. Necht $K \subset P$ je kompaktni. Necht $f : P \to \mathbf{R}$ je spojite zobrazeni na K vzhledem ke K.
Pak $f(k)$ je omezena mnozina v $\mathbf{R}$ (t.j. f je omezena na K).

PROOF. $f(k)$ je kompakt ⇒ omezena. □

THEOREM. *13. (Heine pro limity)*
Necht $(P,\rho), (Q,\sigma)$ jsou metricke prostory, $f : P \to Q, M \subset P$. - nema bejt $M \to Q$ a $M \cap \cup$?
Necht x_0 neni izolovany bod mnoziny M [1] *Pak je ekvivalentni:*
(i) $\lim_{x \to x_0, x \in M} f(x) = y$ (pro nejake $y \in Q$)i
(ii) pro kazdou posloupnost $\{x_n\} \subset M \setminus x_0$, splnujici $x_n \overset{\rho}{\to} x_0$ plati $\lim f(x_n) = y$

PROOF. analogicky jako ve V9 □

NOTE 7.3.7. Z Heineho very lze dokazat VOAL pro funkce $f : (P,\rho) \to \mathbf{R}$
Totez pro V o 2 policajtech

7.3.2. normovane prostory.

DEFINITION 7.3.8. Necht $(X,+,*)$ je vektorovy prostor. Necht o je nulovy prvek X.
normou na prostoru X nazveme zobrazeni $\|.\| : X \to [0,\infty)$ takove, ze
[2]
(i) $\|x\| = 0 \Leftrightarrow x = o$
(ii) $\|\lambda x\| = |\lambda|\,\|x\|\,\forall x \in X \forall\lambda \in \mathbf{T}$
(iii) $\|x+y\| \leq \|x\| + \|y\|\,\forall x,y \in X$

[1] abychom se nezbavili limity v nedefinovanych bodech
[2] v podstate udava absolutni vzdalenost od 0

NOTE 7.3.9. (i) Je-li na X dana norma $\|.\|$, pak lze definovat metriku $\rho(x,y) = \|x-y\|$
(ii) opacne? zkusime def. $\|x\| = \rho(x,o)$
je to norma?
(i) OK
(iii) $\|x+y\| = \rho(x+y,0) \overset{?}{\leq} \rho(x,0)+\rho(y,0)$?
(ii) $\rho(\alpha x, o) \overset{?}{=} \alpha\rho(x,o)$ neplati

EXAMPLE 7.3.10. (i) $(\mathbf{R}, |x-y|)$ lze def. $\|x\| = |x|$
(ii) $\mathbf{R}^n$: $\|x\|_{eukl} = \sqrt{\sum_{i=1}^n |x_i|^2}$ - eukleidovska norma
(iii) $\mathbf{R}^n$ $\|x\|_{l_1} = \sum_{i=1}^n |x_i|$ - souctova norma
(iv) $\mathbf{R}^n$ $\|x\|_{l_\infty} = \max_{i=1,\dots,n} |x_i|$ - maximova norma
(v) $\zeta k([a,b])$: $\|f\|_{\zeta([a,b])} = \sup_{x\in[a,b]} |f(x)|$

7.3.3. normovany prostor. Mnozinu $(X, \|.\|)$ nazveme **normovanym linearnim prostorem**
dimenze X je mohutnost algebraicke baze

EXAMPLE 7.3.11. $\dim(\mathbf{R}^n) = n$
$\dim(\zeta[a,b]) = \infty$

NOTE 7.3.12. Uzavrena jednotkova koule $\overline{B_X}(0,1) = \{y \in X, \|y\| \leq 1\}$ v normovanem linearnim prostoru je kompaktni $\Leftrightarrow \dim X < \infty$
/*quli omezenosti ??*/

CHAPTER 8

Funkce více proměnných

[1]

8.1. limita a spojitost

8.1.1. setup. $\mathbf{R}^n = \underbrace{\mathbf{R} \times \ldots \times \mathbf{R}}_{n}$

$x \in \mathbf{R}^n : x = [x_1, \ldots, x_n]$ (vektor)
na $\mathbf{R}^n$ budeme vzdy uvazovat eukleidovskou metriku a normu
Zajimaji nas tyto objekty: $f : \mathbf{R}^n \to \mathbf{R}$
$f : \mathbf{R}^n \to \mathbf{R}^m, m, n \in \mathbf{N}$
$f : \mathbf{R}^n \to \mathbf{R}^m \Leftrightarrow f = (f_1, \ldots, f_m), f_i = \mathbf{R}^n \to \mathbf{R}, i = 1, \ldots, m$
limita a spojitost techto funkci viz predchozi kapitola

EXAMPLE 8.1.1. $f : \mathbf{R}^n \to \mathbf{R}^m$ je spojita $\Leftrightarrow f_i$ je spojita z $\mathbf{R}^n$ do $\mathbf{R} \forall i = 1, \ldots, m$
"Dukaz" - konvergence v $\mathbf{R}^n$ je konvergence po slozkach
$f : \mathbf{R}^n \to \mathbf{R}^m$ spojita $\overset{Heine}{\Leftrightarrow}$ $[x_n \to x_0 \text{ v } \mathbf{R}^n \Rightarrow f(x_n) \to f(x_0) \text{ v } \mathbf{R}^m]$
$\overset{\text{konvergence po slozkach}}{\Leftrightarrow}$ $[x_n \to x_0 \text{ v } \mathbf{R}^n \Rightarrow (f(x_m))_i \to (f(x_0))_i \text{ v } \mathbf{R} \forall i = 1, \ldots, m]$
$\Leftrightarrow f_i : \mathbf{R}^n \to \mathbf{R}$ jsou spojite $\forall i = 1, \ldots, n$

8.1.2. Vime, ze plati. VOAL (protoze konvergence je konvergence po slozkach)
V o limite a usporadani (totez)
V o 2 policajtech (totez)[2]
V o spojitosti slozene funkce:
obecne $(P, \rho) \overset{f}{\to} (Q, \sigma) \overset{g}{\to} (R, \tau)$
f, g spojite $\Rightarrow (g \bullet f)$ je spojita z (P, ρ) do (R, τ) - nechyby tady cosi?

EXAMPLE 8.1.2. (i) projekce na prvni slozku $f : \mathbf{R}^n \to \mathbf{R}, f : x \to x_1$
Pak f je spojite.
$x^k \to x$ v $\mathbf{R}^n \Leftrightarrow \left[\left(x^k\right)_1, \ldots, \left(x^k\right)_n\right] \to [x_1, \ldots, x_n] \Rightarrow \left(x^k\right)_1 \to x_1$
(ii) $f : \mathbf{R}^n \to \mathbf{R}, f(x) = \|x\| = \sqrt{\sum_{i=1}^n |x_i|^2}$ je spojita
nebot projekce jsou spojite, $|x|^2$ spoj, $\sum$ spoj, $\sqrt{}$spoj
$\Rightarrow$ spojitost funkce plyne z V o spojitosti slozene funkce
(iii) $f(x) = \frac{2xy}{x^2+y^2}$ def. vsude v$\mathbf{R}^2$ krome $[0, 0]$
Lze f dodefinovat v $[0, 0]$ spojite?
$\Rightarrow \exists \lim_{[x,y]\to[0,0]} \frac{2xy}{x^2+y^2}$?

8.1.3. Jak hledame limity.

(1) najdu vhodneho kandidata volbou specialniho smeru: napr $y = kx, y = 0, x = 0$
(2) budto najdu kandidatu vic $\Rightarrow$ limita neexistije, a nebo dokazu z definice, ze limmita $=$ kandidat

EXAMPLE 8.1.3. (i) $f(x) = \frac{2xy}{x^2+y^2}$
dosadim $x = 0 \Rightarrow \lim = 0$
$y = kx : \lim_{x\to0} \frac{2xkx}{x^2+(kx)^2} = \frac{2k}{1+k^2}$
$\Rightarrow$ limita neexistuje a fukce se neda dodefinovat spojite
(ii) $f(x) = \frac{2xy^2}{x^2+y^2}$lze dodefinovat v ?
$y = kx : \lim_{x\to0} \frac{2x^2k^2}{x^2+(kx)^2} = 0$
$x = 0$ taky $\Rightarrow 0$ je dobry kandidat.
$\lim_{[x,y]\to[0,0]} \left|\frac{2xy^2}{x^2+y^2} - 0\right| = \lim_{[x,y]\to[0,0]} |y| \underbrace{\left|\frac{2x}{x^2+y^2}\right|}_{\text{omezene na okoli 0}} \leq C \lim_{[x,y]\to[0,0]} y = 0$
(iii) vsechny smery ($y = kx$)nestaci $f(x, y) = \left\{ \begin{array}{c} 1 : y = x^2 \\ 0 \end{array} \right\}$

[1]hlavni napln tohoto semestru
[2]platila by i v $\mathbf{R}^n$

8.2. parcialni derivace a totalni diferencial

DEFINITION 8.2.1. Necht f je funkce n promennych, $\mathcal{D}(f) \subset \mathbf{R^n}$, necht $x \in \mathcal{D}(d)^0$. //ne hranicni body
Pak **derivaci funkce f podle i-te promenne** ($i = 1, \dots, n$) v bode x rozumime
$\lim_{t\to 0} \frac{f(x_1,\dots,x_i+t,\dots,x_n)-f(x)}{t}$
(klicovy pojem) Tyto derivace nazyvame **parcialnimi derivacemi**, znacime $\frac{\delta f}{\delta x_i}(x)$
($x \in \mathbf{R^n}, x_i \in \mathbf{R}$)
Necht $v \in \mathbf{R^n} \setminus \{0\}$, necht $f : \mathbf{R^n} \to \mathbf{R}, x \in \mathcal{D}(f)^0$.
Derivaci funkce f ve smeru v v bode x rozumime
$\frac{\delta f}{\delta v}(x) = \lim_{t\to 0} \frac{f(x+t(v))-f(x)}{t}$

NOTE 8.2.2. Bazove vektory v $\mathbf{R^n}$: $e_i = \left(0, \dots, 0, \underbrace{1}_{i}, 0, \dots 0\right)$
(i) parcialni derivace $\frac{\delta f}{\delta x_i}(x)$ je derivace ve smeru e_i.
$\frac{\delta f}{\delta x_i}(x) = \frac{\delta f}{\delta e_i}(x)$
(ii) Jestlize $\exists$ vsechny parcialni derivace $\frac{\delta f}{\delta x_i}(x), i = 1, \dots, n$, pak nemusi nutne existovat $\frac{\delta f}{\delta v} \forall v \in \mathbf{R^n}$
def $f(x) = \begin{cases} 1, xy = 0 \\ 0 \end{cases}$
Pak $\exists \frac{\delta f}{\delta x}(0,0), \frac{\delta f}{\delta y}(0,0)$, ale neexistuje napr. $\frac{\delta f}{\delta v}(0,0), v = [1,1]$
(iii) Z $\exists \frac{\delta f}{\delta v} \forall v \in \mathbf{R^n}$ **neplyne** f v x spojita
graf funkce $y = x^2$ bez nuly
$f(x) = \begin{cases} 1, y = x^2, x \neq 0 \\ 0 \end{cases}$ MA28
Pak $\exists \frac{\delta f}{\delta v}(0,0) \forall v$
ale f neni v $(0,0)$ spojita
(iv) neplati linearita derivace ve smeru ($\frac{\delta f}{\delta(v_1+v_2)} \neq \frac{\delta f}{\delta v_1} + \frac{\delta f}{\delta v_2}$)
(v) homogenita - pro $\alpha \neq 0 \in \mathbf{R}$ plati $\frac{\delta f}{\delta(\alpha v)} = \alpha \frac{\delta f}{\delta v}$ (CVICENI)

EXAMPLE 8.2.3. $f(x,y) = e^x + xy^2$, spoctete $\frac{\delta f}{\delta x}, \frac{\delta f}{\delta y}$
$\frac{\delta f}{\delta x}(x,y) = e^x + y^2$
$\frac{\delta f}{\delta y}(x,y) = 0 + x2y = 2xy$

8.2.1. motivace pro zavedeni totalniho defirencialu. $f : \mathbf{R} \to \mathbf{R}, a \in \mathbf{R}$, necht $\exists f\prime(a)$
$f(a+h) \approx f(a) + f\prime(a)h$ - nejlepsi linearni aproximace
Definujme $L(h) = f\prime(a)h$ (linearni zobrazeni $\mathbf{R} \to \mathbf{R}$)
potom plati: $\lim_{h\to 0} \frac{f(a+h)-f(a)-L(h)}{h} = 0$ (viz V o aproximaci Taylorovym polynomem)

DEFINITION 8.2.4. **(klicovy pojem)** Necht f je funkce n promennych, $f : \mathbf{R^n} \to \mathbf{R}$. Rekneme, ze linearni zobrazeni $L : \mathbf{R^n} \to \mathbf{R}$ je **totalnim diferencialem** funkce f v bode a, jestlize $\lim_{h\to o} \frac{f(a+h)-f(a)-L(h)}{\|h\|} = 0$
znaceni: Totalni diferencial f v bode a je linearni zobrazeni, ktere znacime
$Df(a)$ (zobrazeni $\mathbf{R^n} \to \mathbf{R}$)
$Df(a)(h)$ hodnota zobrazeni v "bode" h

NOTE 8.2.5. (i) Limitu v definici totalniho diferencialu lze ekvivalentne prepsat:
$\lim_{x\to a} \frac{f(x)-f(a)-L(x-a)}{\|x-a\|}$
(ii) f ma v bode a totalnih diferencial $L(h) \Leftrightarrow$ funkce $\eta, \eta(h) = f(a+h) - f(a) - L(h)$ splnuje $\lim_{h\to o} \frac{\eta(h)}{\|h\|} = 0$

THEOREM. *1. (tvar totalniho diferencialu) LV*
Necht f je funkce n promennych ($f : \mathbf{R^n} \to \mathbf{R}$), necht $a \in \mathbf{R^n}$.
Necht f ma v a totalni diferencial $Df(a)$.
Potom $\exists$vlastni derivace f ve vsech smerech a plati: $\frac{\delta f}{\delta v}(a) = Df(a)(v), v \in \mathbf{R^n} \setminus \{o\}$.
Navic $\forall h = [h_1, \dots, h_n] \in \mathbf{R^n} : Df(a)(h) = \frac{\delta f}{\delta x_1}(a)h_1 + \frac{\delta f}{\delta x_2}(a)h_2 + \dots + \frac{\delta f}{\delta x_n}(a)h_n$.

PROOF. Vime, ze $\lim_{h\to o} \frac{f(a+h)-f(a)-Df(a)(h)}{\|h\|} = 0$.
Zvolme $v \neq o \in \mathbf{R^n}$. Potom specialne pro $h = tv, t \in \mathbf{R}$
$\lim_{t\to 0} \frac{f(a+tv)-f(a)-Df(a)(tv)}{\|tv\|} = 0$
$Df(a)$ je linearni $\Rightarrow Df(a)(tv) = tDf(a)(v)$. Tedy
$\frac{1}{\|v\|} \lim_{t\to 0} \left(\frac{f(a+tv)-f(a)}{|t|} - Df(a)(v)\right) = 0 \Rightarrow \frac{\delta f}{\delta v}(a) = Df(a)(v)$
Dale vime, ze $Df(a)$ je linearni zobrazeni z $\mathbf{R^n}$ do $\mathbf{R}$. Tedy $Df(a)(h) = \sum_{i=1}^n A_i h_i, A_i \in \mathbf{R}$
Z prave dokazaneho tvrzeni $\frac{\delta f}{\delta v} = Df(v)$ puziteho na $v = e_i$ mame:
$\frac{\delta f}{\delta x_i}(a) = \frac{\delta f}{\delta e_i}(a) = Df(a)(e_i) = A_1 0 + \dots + A_i 1 + A_{i+1} 0 + \dots + A_n 0 = A_i$
Tedy $A_i = \frac{\delta f}{\delta x_i}(a)$. $\square$

8.2.2. gradient.

NOTE 8.2.6. Na $\mathbf{R^n}$ je definovan skalarni soucin: $x, y \in \mathbf{R^n}, x \cdot y = \sum_{i=1}^{n} x_i y_i$
Tedy prave dokazana veta rika, ze $Df(a)(h) = \left(\frac{\delta f}{\delta x_1}(a), \ldots, \frac{\delta f}{\delta x_n}(a)\right) \cdot h$

DEFINITION 8.2.7. Necht f ma v bode a totalni diferencial. Vektor $\left(\frac{\delta f}{\delta x_1}(a), \ldots, \frac{\delta f}{\delta x_n}(a)\right)$ nazyvame **gradientem f v bode** a a znacime (nabla) $\nabla f(a)$

NOTE 8.2.8. Gradient ma vyznam algebraicky,ale i geometricky:
$\frac{\delta f}{\delta v}(a) = Df(a)(v) = \nabla f(a) \cdot v = \sum_{i=1}^{n} \frac{\delta f}{\delta x_i}(a) v_i$
Predpokladejme, ze $\|v\| = 1$. Pak mame $\overset{Cauchy}{\leq} \sqrt{\sum_{i=1}^{n} \left(\frac{\delta f}{\delta x_i}(a)\right)^2} \sqrt{\sum_{i=1}^{n} v_i^2} = \|\nabla f(a)\| \|v\| = \|\nabla f(a)\|$
rovnost plati $\Leftrightarrow \nabla f(a) = kv$ (pro nejake $k \geq 0$) (CVICENI)
To znamena, ze gradient udava smer nejvetsiho spadu

THEOREM. *2. (vztah spojitosti a totalniho diferencialu) LV*
Necht $f : \mathbf{R^n} \to \mathbf{R}$ ma v bode a totalni diferencial.
Potom je f v bode a spojita.

PROOF. $f(a+h) - f(a) = \eta(h) + Df(a)(h)$
$\lim_{h \to o} f(a+h) - f(a) = \underbrace{\lim_{h \to o} \frac{\eta(h)}{\|h\|} \|h\|}_{0-poznamky} + \underbrace{\lim_{h \to o} Df(a)(h)}_{0\text{(linearni zobrazeni)}}$ □

NOTE 8.2.9. $|x \cdot y| \leq \|x\| \|y\|$
nebot
$\left|\sum_{j=1}^{n} x_j y_j\right| \overset{Cauchy}{\leq} \sqrt{\sum_{j=1}^{n} x_j^2} \sqrt{\sum_{j=1}^{n} y_j^2} = \|x\| \|y\|$

dusledek: Ma-li f v a totalni diferencial, pak $\forall h \in \mathbf{R^n}$
$|Df(a)(h)| = |\nabla f(a) \cdot f| \leq \|\nabla f(a)\| \|h\|$

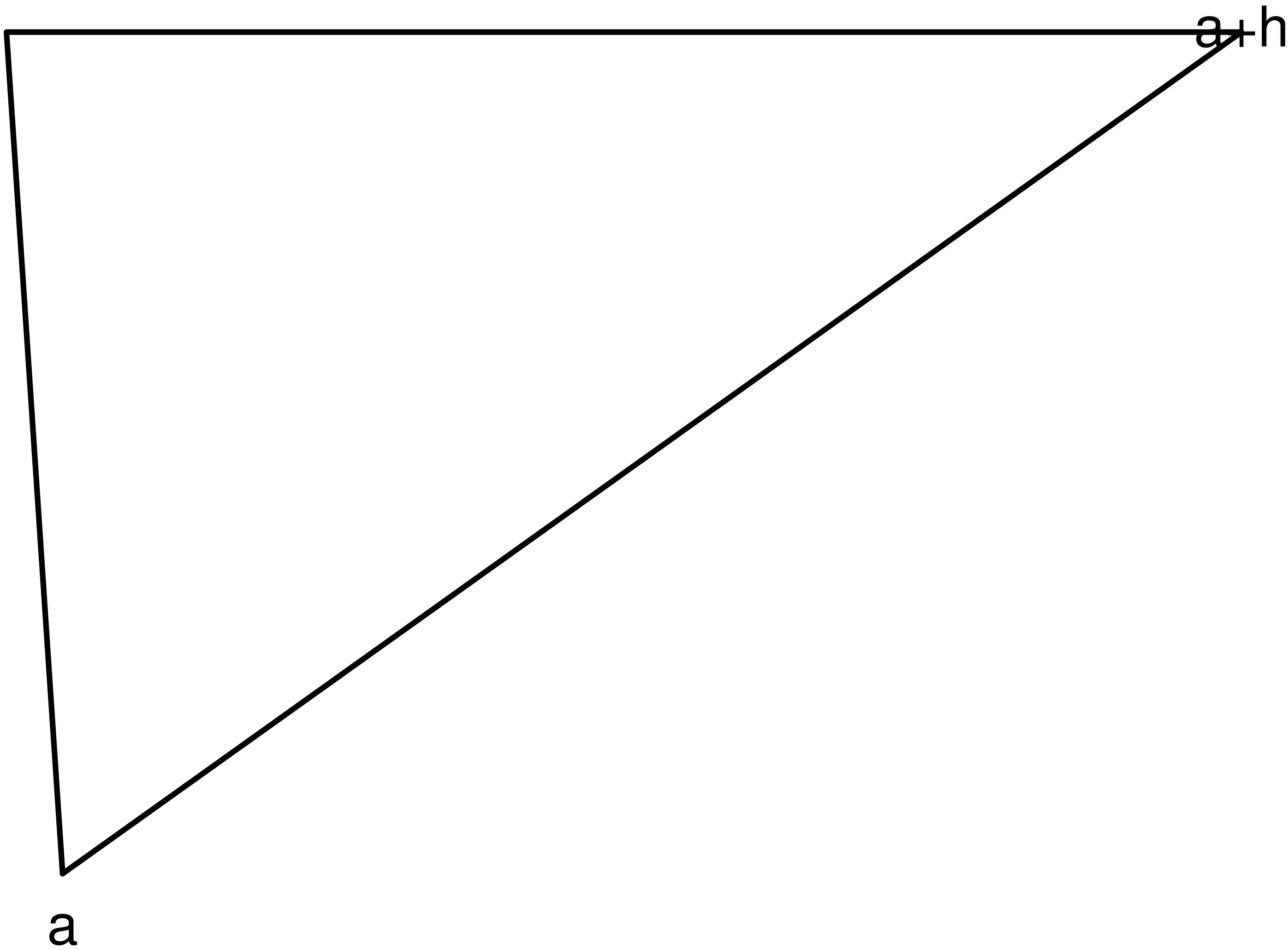
THEOREM. *3. (aritmetika totalniho diferencialu) LV*
Necht $f : \mathbf{R^n} \to \mathbf{R}, g : \mathbf{R^n} \to \mathbf{R}$, necht $\exists Df(a), Dg(a), a \in \mathbf{R^n}, c \in \mathbf{R}$
Potom $\exists$ take $D(f+g)(a), D(cf)(a), D(fg)(a)$
a jestlize navic $g(a) \neq 0$, tak $\exists$ take $D\left(\frac{f}{g}\right)(a)$ a plati:
(i) $D(f+g)(a) = Df(a) + Dg(a)$
(ii) $D(cf)(a) = cDf(a)$
(iii) $D(fg)(a) = g(a) Df(a) + f(a) Dg(a)$
(iv) $D\left(\frac{f}{g}\right)(a) = \frac{g(a)Df(a) - f(a)Dg(a)}{(g(a))^2}$

PROOF. Dokazuji pouze (iii) - neprijemnej zapis
Musime dokazat:
$\lim_{h \to o} \frac{1}{\|h\|} (f(a+h) g(a+h) - f(a) g(a) - g(a) Df(a)(h) - f(a) Dg(a)(h)) = 0$
pucit a vratit:
mame
$= f(a+h) g(a+h) - f(a) g(a) - g(a) Df(a)(h) - f(a) Dg(a)(h)$
$= g(a)[f(a+h) - f(a) - Df(a)(h)] + f(a+h)[g(a+h) - g(a)] - f(a) Dg(a)(h)$
$= g(a)[f(a+h) - f(a) - Df(a)(h)] + f(a+h)[g(a+h) - g(a) - f(a) Dg(a)(h)] + Dg(a)(h)[f(a+h) - f(a)]$
dale mame:
$\lim_{h \to o} g(a) \frac{[f(a+h) - f(a) - Df(a)(h)]}{\|h\|} = 0 \; (g(a) < \infty, \exists Df(a))$
$\lim_{h \to o} f(a+h) \frac{[g(a+h) - g(a) - f(a) Dg(a)(h)]}{\|h\|} = (\exists Df(a) \Rightarrow f$ v a spojita)
$\overset{VOAL}{=} \lim_{h \to o} f(a) \frac{[g(a+h) - g(a) - f(a) Dg(a)(h)]}{\|h\|} = 0 \; (f(a) < \infty, \exists Dg(a))$
zbyva
$\lim_{h \to o} \underbrace{\frac{Dg(a)(h)}{\|h\|}}_{\text{omezene(dusledek)}} \underbrace{(f(a+h) - f(a))}_{\to 0 (f \text{ spoj})}$ □

NOTE 8.2.10. f ma v a totalni diferencial $\Rightarrow \exists$ parcicalni derivace $\exists \frac{\delta f}{\delta x_i}(a)$, $\exists \frac{\delta f}{\delta v} \forall v \neq 0 \in \mathbf{R^n}$, f je spojita a a hledame postacujici podminku pro totalni diferencial

THEOREM. *4. (postacijici podminka pro existenci totalniho diferencialu) TV*
Necht $f : \mathbf{R^n} \to \mathbf{R}, a \in \mathbf{R^n}$, Necht funkce $\frac{\delta f}{\delta x_j} : \mathbf{R^n} \to \mathbf{R}, j = 1, \ldots, n$ jsou spojite v a (t.j. funkce $x \to \frac{\delta f}{\delta x_j}(x)$ jsou spojite v a $\forall j$)
Potom f ma v a totalni diferencial.

Proof. /*rozeslozkovat - $g_j(t) = f(a_1 + h_1, \ldots, a_j + th_j, a_{j+1}, \ldots, a_n)$
$(\Rightarrow g_j(0) = A_{j-1} \& g_j(1) = A_j, A_j = (a_1 + h_1, \ldots, a_j + h_j, a_{j+1}, \ldots, a_n), A_n = a + h)$
Lagrange pro vsechny slozkovy funkce, 3uhelnik & spojitost p.d.*/



posunuju se po jedny souradny ose, najdu ξ
$f(a+h) - f(a) = \sum_{j=1}^{n} (f(a_1 + h_1, \ldots, a_j + h_j, a_{j+1}, a_n) - f(a_1 + h_1, \ldots, a_{j-1} + h_{j-1}, a_j, \ldots, a_n))$
Lagrange pouzita na funkce g_i definovane:
$g_j(t) = f(a_1 + h_1, \ldots, a_j + th_j, a_{j+1}, \ldots, a_n)$
$g_j(0) = f(A_{j-1})$, $g_j(1) = f(A_j)$, g_j spojita na $(0,1)$ a ma tam derivaci (nebot $\exists \frac{\delta f}{\delta x_j}$)
Tedy $\exists \xi^j$ tak, ze $g_j(1) - g_j(0) = g_j\prime(\xi^j) = \frac{\delta f}{\delta x_i}(\xi^j)$
(ξ^j lezi na useckach $\overline{A_{j-1}, A_j}$, $A_j = (a_1 + h_1, \ldots, a_j + h_j, a_{j+1}, \ldots, a_n)$)
tak, ze $f(a+h) - f(a) = \sum_{j=1}^{n} \frac{\delta f}{\delta x_j}(a) h_j$, kde $\xi^j \in \overline{A_{j-1}A_j}$
Tedy $\frac{f(a+h) - f(a) - \sum_{j=1}^{n} \frac{\delta f}{\delta x_j}(a) h_j}{\|h\|} = \frac{1}{\|h\|}\left(\sum_{j=1}^{n} \frac{\delta f}{\delta x_j}(\xi^j) h_j - \sum_{j=1}^{n} \frac{\delta f}{\delta x_j}(a) h_j\right)$
$= \frac{1}{\|h\|} \sum_{j=1}^{n} \left(\frac{\delta f}{\delta x_j}(\xi^j) - \frac{\delta f}{\delta x_j}(a)\right) h_j$
Tudiz $\frac{\left|f(a+h) - f(a) - \sum_{j=1}^{n} \frac{\delta f}{\delta x_j}(a) h_j\right|}{\|h\|} \overset{\text{trojuhel.}}{\leq} \frac{1}{\|h\|} \sum_{j=1}^{n} \left|\frac{\delta f}{\delta x_j}(\xi^j) - \frac{\delta f}{\delta x_j}(a)\right| \|h\| \to 0$(spojitost parc. derivaci)
$\Rightarrow$ funkce ma totalni diferencial □

Note 8.2.11. Ani tuto implikaci nelze obratit

Lemma. *Necht $g_j, j = 1, \ldots, n$ jsou jako ve vete 5 ($\exists D g_j(a)$)*
Potom $\exists c > 0, \varepsilon > 0 \forall h \in B(o, \varepsilon) : \|g(a+h) - g(a)\|_{\mathbf{R^n}} \leq c\|h\|_{\mathbf{R^s}}$

Proof. $\|g(a+h) - g(a)\| \leq \sum_{j=1}^{n} |g_j(a+h) - g_j(a)|$
Pro kazde g_j platí:
$\frac{g_j(a+h) - g_j(a) - \nabla g_j(a) \cdot h}{\|h\|} \to 0$
/*$\lim_{h \to o} \frac{g_j(a+h) - g_j(a) - \nabla g_j(a) \cdot h}{\|h\|} = \lim_{h \to o} \frac{g_j(a+h) - g_j(a)}{\|h\|} - \lim_{h \to o} \frac{\nabla g_j(a) \cdot h}{\|h\|} = 0$
je to okolo $o \Rightarrow$ muze se to psat normama
$\frac{|g_j(a+h) - g_j(a)|}{\|h\|} - \frac{|\nabla g_j(a) \cdot h|}{\|h\|} \leq ?$
$|g_j(a+h) - g_j(a)| \leq \|\nabla g_j(a) \cdot h\| \|h\|$
*/

Tedy $\exists\varepsilon_j : \forall h \in B(o,\varepsilon_j) : |g_j(a+h) - g_j(a)| \leq \|h\| + \|\nabla g_j(a)\| \|h\|$
Nyni staci vzit $\varepsilon = \min\{\varepsilon_j, j = 1,\ldots,n\}, c = \sum_{j=1}^{n} (\|\nabla g_j(a)\| + 1)$ □

THEOREM. 5. *(diferencial slozeneho zobrazeni) TV*
$n, s \in \mathbf{N}$
$(\mathbf{R^s} \overset{g}{\rightarrow} \mathbf{R^n} \overset{f}{\rightarrow} \mathbf{R})$
$g : \mathbf{R^s} \rightarrow \mathbf{R^n}$, *tedy* $g = (g_1,\ldots,g_n), g_i : \mathbf{R^s} \rightarrow \mathbf{R}$
Necht $H = f \bullet g$, *tedy* $H : \mathbf{R^s} \rightarrow \mathbf{R}$, $x \in \mathbf{R^s}, H(x) = f(g_1(x),\ldots g_n(x))$
Necht $a \in \mathbf{R^s}, b \in \mathbf{R^n}, g_j(a) = b_j$.
Necht $\exists Df(b), \exists Dg_j(a) \forall j = 1,\ldots,n$
Potom $H = f \bullet g$ *ma totalni diferencial v bode a, a plati*
$DH(a)(h) = \sum_{i=1}^{s} \left[\sum_{j=1}^{n} \frac{\delta f}{\delta y_j}(b) \frac{\delta g_j}{\delta x_i}(a)\right] h_i$ */*parcialni derivace δg_j lze chapat jako matici a δf jako vektor*/*

$$\nabla H(a) = \begin{pmatrix} \frac{\delta f}{\delta y_1} \\ \vdots \\ \frac{\delta f}{\delta y_n} \end{pmatrix} \begin{pmatrix} \frac{\delta g_1}{\delta x_1} & \cdots & \frac{\delta g_n}{\delta x_1} \\ \vdots & \ddots & \vdots \\ \frac{\delta g_1}{\delta x_s} & \cdots & \frac{\delta g_n}{\delta x_s} \end{pmatrix} = \left(\sum_{j=1}^{n} \frac{\delta f}{\delta y_j}(b) \frac{\delta g_j}{\delta x_1}(a), \ldots, \sum_{j=1}^{n} \frac{\delta f}{\delta y_j}(b) \frac{\delta g_j}{\delta x_n}(a)\right)$$

(pripomenme: $(f \bullet g)'(a) = f'(g(a))g'(a)$)

PROOF. /*prasacky pocitani
def $\eta(u) = f(b+u) - f(b) - Df(b)(u)$
def $\psi_j(h) = g_j(a+h) - g_j(a) - Dg_j(a)(h)$
def $\tau(h) = H(a+h) - H(a) - L(h)$
a pocitat a pocitat ... pouziti Lemmatu $\exists c > 0, \varepsilon > 0 \forall h \in B(o,\varepsilon) : \|g(a+h) - g(a)\|_{\mathbf{R^n}} \leq c\|h\|_{\mathbf{R^s}}$*/
(neni v podstate tezkej ¯ zapis toho, co mame)
Polozime $\eta(u) = f(b+u) - f(b) - Df(b)(u)$ $(u \in \mathbf{R^n})$
$\psi_j(h) = g_j(a+h) - g_j(a) - Dg_j(a)(h)$ $(h \in \mathbf{R^s})$ $j = 1,\ldots,n$
Potom z $\exists$ diferenciálů vime:
$\lim_{u\rightarrow o} \in \mathbf{R^n} \frac{\eta(u)}{\|u\|} = 0, \lim_{h\rightarrow o \in \mathbf{R^s}} \frac{\psi_j(u)}{\|u\|} = 0 \forall j = 1,\ldots,n$
Poloz $L(h) = \sum_{i=1}^{s} \left[\sum_{j=1}^{n} \frac{\delta f}{\delta y_j}(b) \frac{\delta g_j}{\delta x_i}(a)\right] h_i$,
$\tau(h) = H(a+h) - H(a) - L(h)$
Chceme $\lim_{h\rightarrow o \in \mathbf{R^s}} \frac{\tau(h)}{\|h\|_{\mathbf{R^s}}}$
dosadim definice:
$\tau(h) = f(g(a+h)) - f(g(a)) - L(h)$
jedinny trik: $= f\left(\underbrace{g(a)}_{b} + \underbrace{g(a+h) - g(a)}_{u}\right) - f\left(\underbrace{g(a)}_{b}\right) - L(h)$ $(g(a) = b)$
$= \eta(g(a+h) - g(a)) + Df(b)(g(a+h) - g(a)) - L(h)$ (definice η pro $b = g(a), u = g(a+h) - g(a)$)
clen $Df(b)(g(a+h) - g(a)) = \nabla f(b) \cdot (g(a+h) - g(a))$
$= \sum_{i=1}^{n} \frac{\delta f}{\delta y_j}(b)(g_j(a+h) - g_j(a))$ (def. gradientu a skalarniho soucinu)
$= \sum_{i=1}^{n} \frac{\delta f}{\delta y_j}(b)(\psi_j(h) + Dg_j(a)(h))$ (def. ψ_j)
$= \sum_{i=1}^{n} \frac{\delta f}{\delta y_j}(b)\psi_j(h) + \sum_{i=1}^{n} \frac{\delta f}{\delta y_j}(b) \sum_{i=1}^{s} \frac{\delta g_j}{\delta x_i}(a) h_i$ (def ∇ a skalarniho soucinu)
$= \sum_{i=1}^{n} \frac{\delta f}{\delta y_j}(b)\psi_j(h) + L(h)$
Tedy $\tau(h) = \eta(g(a+h) - g(a)) + \sum_{j=1}^{n} \frac{\delta f}{\delta y_j}(b)\psi_j(h)$

$$\Rightarrow \frac{\tau(h)}{\|h\|} = \frac{\eta(g(a+h)-g(a))}{\|h\|} + \underbrace{\sum_{j=1}^{n} \frac{\delta f}{\delta y_j}(b) \underbrace{\frac{\psi_j(h)}{\|h\|}}_{\rightarrow 0 \text{ (vlastnost } \psi_j)}}_{\rightarrow 0} \quad \left(\begin{matrix} h \rightarrow o \\ \rightarrow \end{matrix} \; 0?\right)$$

Zbyva: $\lim_{h\rightarrow o \in \mathbf{R^s}} \frac{\eta(g(a+h)-g(a))}{\|h\|} = 0$
Jestlize $g(a+h) = g(a)$, neni co dokazovat
$g(a+h) - g(a) \rightarrow o \in \mathbf{R^n}$ nebot g je spojita
Jinak $\underbrace{\frac{\eta(g(a+h) - g(a))}{\|g(a+h) - g(a)\|_{\mathbf{R^n}}}}_{\rightarrow 0} \underbrace{\frac{\|g(a+h) - g(a)\|_{\mathbf{R^n}}}{\|h\|_{\mathbf{R^s}}}}_{\text{omezene diky Lemmatu}} \rightarrow 0$ □

EXAMPLE 8.2.12. $f : \mathbf{R}^2 \rightarrow \mathbf{R}$
def $F(x) = f(\sin x, \cos x), F : \mathbf{R} \rightarrow \mathbf{R}$
(parametrizovani krivky v $\mathbf{R}^2$)
chci spocitat $F'(x)$
$\mathbf{R} \rightarrow^{g} \mathbf{R}^2 \rightarrow^{f} \mathbf{R}$, $\longrightarrow^{F}$
$F'(x) = \frac{\delta f}{\delta x}(\sin x, \cos x) g_1'(x) + \frac{\delta f}{\delta x}(\sin x, \cos x) g_2'(x) = \cos x \frac{\delta f}{\delta x}(\sin x, \cos x) - \sin x \frac{\delta f}{\delta y}(\sin x, \cos x)$

Dusledek: (retizkove pravidlo - chain rule). $\mathbf{R^s} \overset{g}{\to} \mathbf{R^n} \overset{f}{\to} \mathbf{R}$, $\exists$ diferencialy

pak:
$\frac{\delta(f\bullet g)}{\delta x_i}(a) = \sum_{j=1}^{n} \frac{\delta f}{\delta y_j}(g(a)) \frac{\delta g_j}{\delta x_i}(a)$ pro $i = 1, \ldots, s$

THEOREM. *6. (stredni hodnote pro funkce vice promennych) LV*
Necht f je funkce n promennych, $a, b \in \mathbf{R^n}$, necht f ma v kazdem bode usecky $\overline{ab} = \{y \in \mathbf{R^n}; \exists t \in (0,1) : y = a + t(b-a)\}$ totalni diferencial.
Potom $\exists \xi \in (0,1) : f(b) - f(a) = \sum_{j=1}^{n} \frac{\delta f}{\delta x_j}(a + \xi(b-a))(b_j - a_j)$
/$= Df(a + \xi(b-a))(b-a)$*/*
*/*existuje bod, ve kterem gradient udava presne spad linearnimu posunu od a do b*/*

PROOF. (prevedeni na Lagrangovu)
def $F(t) = f(a + t(b-a)), t \in [0,1]$
Potom $F(0) = f(a), F(1) = f(b)$
$F\prime(t)$: pouzijeme V5: $\underbrace{[0,1] \to^{g} \mathbf{R^n} \to^{f} \mathbf{R}}_{\to F}$
def $g_j(t) = a_j + t(b_j - a_j)$
(potom zrejme $F(t) = f(g(t))$)
Tedy podle chain rule:
$F\prime(t) = \sum_{j=1}^{n} \frac{\delta f}{\delta x_j}(g(t)) g_j\prime(t) = \sum_{j=1}^{n} \frac{\delta f}{\delta x_j}(a + t(b-a))(b_j - a_j)$
Tedy F ma derivaci v kazdem bode $t \in [0,1]$, je spojita (slozeni spojitych zobrazeni)
A tedy podle Lagrange: $\exists \xi \in (0,1) : F(1) - F(0) = F\prime(\xi)1$
Dosadime: $f(b) - f(a) = \sum_{j=1}^{n} \frac{\delta f}{\delta x_j}(a + \xi(b-a))(b_j - a_j)$ □

8.3. parcialni derivace a diferencialy vyssich radu

$f : \mathbf{R^n} \to \mathbf{R}$, necht $\exists \frac{\delta f}{\delta x_i}$ (vsude) $\forall i = 1, \ldots, n$

potom $\frac{\delta f}{\delta x_i} : \mathbf{R^n} \to \mathbf{R}$, tedy lze uvazovat o dalsim derivovani - $\frac{\delta\left(\frac{\delta f}{\delta x_i}\right)}{\delta x_j}$ (druha derivace)

DEFINITION 8.3.1. **Druhou parcialni derivaci** funkce f v bode a podle promennych x_i, x_j nazyvame parcialni derivaci $\frac{\delta f}{\delta x_i}$ podle promenne x_j, ma-li tato derivace smysl.
Znacime: $\frac{\delta^2 f}{\delta x_i \delta x_j} = \frac{\delta}{\delta x_j}\left(\frac{\delta f}{\delta x_i}\right)$, jestlize $i = j$, pak znacime $\frac{\delta^2 f}{\delta x_i^2}$

EXAMPLE 8.3.2. $f(x,y) = x^y$ $(x > 0)$
$\frac{\delta f}{\delta x}(x,y) = yx^{y-1}$
$\frac{\delta f}{\delta y}(x,y) = x^y \log x$
$\frac{\delta^2 f}{\delta^2 x}(x,y) = \frac{\delta\left(yx^{y-1}\right)}{\delta x} = y(y-1)x^{y-2}$
$\frac{\delta^2 f}{\delta x \delta y}(x,y) = \frac{\delta\left(yx^{y-1}\right)}{\delta y} = x^{y-1} + yx^{y-1}\log x$
$\frac{\delta^2 f}{\delta y^2}(x,y) = \frac{\delta(x^y \log x)}{\delta y} = x^y(\log x)^2$
$\frac{\delta^2 f}{\delta y \delta x}(x,y) = \frac{\delta(x^y \log x)}{\delta x} = yx^{y-1}\log x + x^y\frac{1}{x} = yx^{y-1}\log x + x^{y-1}$
hle! $\frac{\delta^2 f}{\delta x \delta y}(x,y) = \frac{\delta^2 f}{\delta y \delta x}(x,y)$ /*inzenyri veri, ze to plati uplne vzdycky*/

THEOREM. *7. (symetrie smisenych derivaci) BEZ DUKAZU*
Necht $f : \mathbf{R^n} \to \mathbf{R}$ ma v bode $a \in \mathbf{R^n}$ spojitou derivaci $\frac{\delta^2 f}{\delta x_i \delta x_j}$, kde $i, j \in \{1, \ldots, n\}$
Potom $\exists$ take $\frac{\delta^2 f}{\delta x_j \delta x_i}$ a $\frac{\delta^2 f}{\delta x_i \delta x_j} = \frac{\delta^2 f}{\delta x_j \delta x_i}$

DEFINITION 8.3.3. Rekneme, ze $f : \mathbf{R^n} \to \mathbf{R}$ ma v bode a **druhy (totalni) diferencial** $D^2 f(a)$, jestlize kazda parcialni derivace $\frac{\delta f}{\delta x_i}, i = 1, \ldots, n$ ma totalni diferencial v bode a.
Druhy diferencial je linearni zobrazeni z $\mathbf{R^n} \times \mathbf{R^n}$ do $\mathbf{R}$ a ma tvar
$D^2 f(a)(h,k) = \sum_{i,j=1}^{n} \frac{\delta^2 f}{\delta x_i \delta x_j}(a) h_i k_j$
Je to bilinearni forma z $\mathbf{R^n} \times \mathbf{R^n}$ do $\mathbf{R}$
$A(h \cdot k)$, lin. $A(h, \cdot), A(\cdot, k)$

NOTE 8.3.4. Prvni diferencial reprezentujeme vektorem $\left(\frac{\delta f}{\delta x_1}(a), \ldots, \frac{\delta f}{\delta x_n}(a)\right)$

THEOREM. *8˙ (tvar druheho diferencialu)*
Jestlize f ma v $a \in \mathbf{R^n}$ druhy diferencial,
pak $D^2 f(a)$ ma tvar $D^2 f(a)(h \cdot k) = \sum_{i,j=1}^{n} \frac{\delta^2 f}{\delta x_i \delta x_j}(a) h_i k_j$
*a reprezentujici matice $\left(\frac{\delta f}{\delta x_i \delta x_j}(a)\right)_{i=1,j=1}^{n}$ je symetricka /*neznamena to, ze $\frac{\delta^2 f}{\delta x_i x_j} = \frac{\delta^2 f}{\delta x_j x_i}$?? */*
BEZ DUKAZU

DEFINITION 8.3.5. Necht $G \subset \mathbf{R}^{\mathbf{n}}$ je otevrena. Pak znacime -proc open?
$\mathcal{C}(G) = \{f : G \to \mathbf{R} \text{ spojite}\}$
$\mathcal{C}^1(G) = \left\{f : G \to \mathbf{R}, \frac{\delta f}{\delta x_i} : G \to \mathbf{R} \text{ jsou spojite} \forall i = 1, \ldots, n\right\}$
$\mathcal{C}^2(G) = \left\{f : G \to \mathbf{R}, \frac{\delta^2 f}{\delta x_i \delta x_j} : G \to \mathbf{R} \text{ jsou spojite} \forall i, j \ldots\right\}$
$\mathcal{C}^k(G) = \{\text{funkce se spojitymi parcialnimi derivacemi az do radu } k\}$
$\mathcal{C}^\infty(G) = \bigcap_{k=1}^{\infty} \mathcal{C}^{\|}(\mathcal{G})$

NOTE 8.3.6. (i) $\mathcal{C}$ se nekdy znaci $\mathcal{C}^0$
(ii) $\mathcal{C}^\infty(G) \subset \mathcal{C}^k(G) \subset \mathcal{C}^l(G) \subset \mathcal{C}^0(G)\, \forall 0 < l \le k < \infty$
(iii) polynomy $\in \mathcal{C}^\infty(\mathbf{R}^{\mathbf{n}})$
$\sin, \cos \in \mathcal{C}^\infty(\mathbf{R}^{\mathbf{n}})$
$e^{-\|x\|^2} \in \mathcal{C}^\infty(\mathbf{R}^{\mathbf{n}})$

THEOREM. *9.*
Necht $f \in \mathcal{C}^2(G)$, kde $G \subset \mathbf{R}^{\mathbf{n}}$ je otevrena.
Potom f ma druhy diferencial v kazdem bode $a \in G$
*/*dukaz neni zapotrebi - je jednoduchej*/*

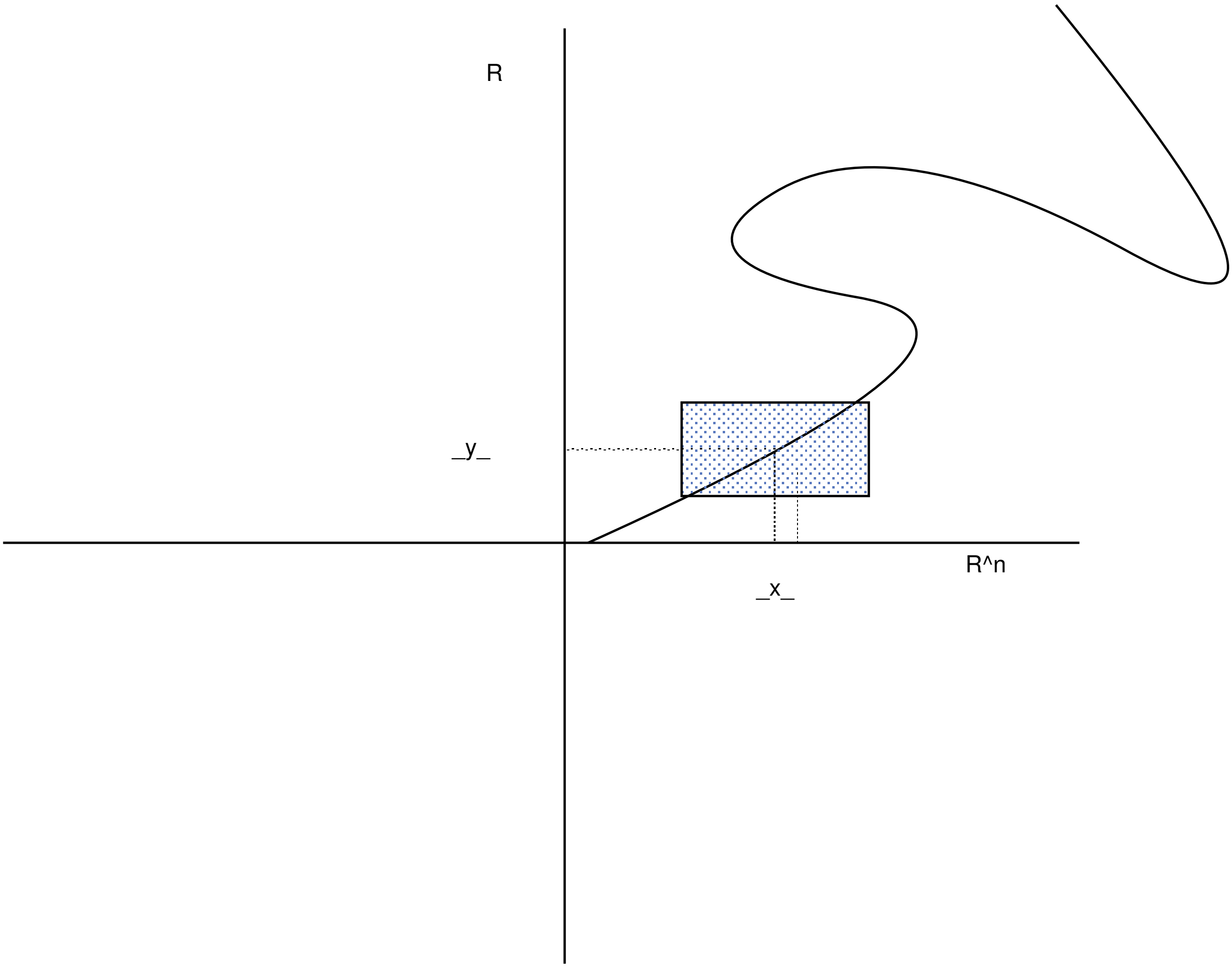
8.4. implicitni funkce

v pisemce aspon jeden priklad dycky
plocha kruhu. $kruh = \{[x, y] : x^2 + y^2 \le 1\}$
umim Riemannovym integralem, ale musim si vyjadrit jednu promennou
$y = \sqrt{1 - x^2}$
$y \ldots$zavisla promenna
$x \ldots$nezavisla promenna
tedy $y = y(x)$ byla zadana implicitne podminkou $x^2 + y^2 = 1$
vice promennych. $F(x, y, z) = 0$ $z = z(x, y)$
stavova rovnice pro idealni plyn. $PV = nRT$
$F(P, V, T) = PV - nRT = 0$
kterakoli promenna je funkci tech zbyvajicich promennych
vetsinou potrebuju derivaci
energie nezavisi na teplote $(\frac{\delta E}{\delta T} = 0)$
$P = P(V, T)$
zajimaji me $\frac{\delta P}{\delta T}, \frac{\delta P}{\delta V}$- neni potreba vyjadreni elementarnimi funkcemi, ale treba v jednom bode
sfera. $x^2 + y^2 + z^2 = 0$
kde muzeme rozvinout z jako funkci x a y ?
- mimo rovnik
myslenka. Necht $F(x, y) = 0, y = y(x), \exists y\prime(x)$
derivuju $\frac{\delta}{\delta x}(F(x, y(x)) = 0)$
??? podle chain rule: $\frac{\delta F}{\delta x}(x, y) \underbrace{\frac{\delta x}{\delta x}}_{1} + \frac{\delta F}{\delta y}(x, y)\, y\prime(x) = 0$
kdy mohu napsat $y\prime(x) = -\frac{\frac{\delta F}{\delta x}(x,y)}{\frac{\delta F}{\delta y}(x,y)}$?
$\frac{\delta F}{\delta y}(x, y) \ne 0$ ($\Rightarrow$ rovnik)
$F(x, y, z) = 0$ chci $\frac{\delta z}{\delta x}(x, y)$
$\frac{\delta F}{\delta x}(x, y, z(x, y)) = \frac{\delta F}{\delta x}(x, y, z) \underbrace{\frac{dx}{dx}(x)}_{1} + \underbrace{\frac{\delta F}{\delta y}(x, y, z) \frac{dx}{dx}(x)}_{0} + \frac{\delta F}{\delta z}(x, y, z) \frac{dz}{dx}(x, y) = 0$
$\Rightarrow \frac{\delta z}{\delta x}(x, y) = -\frac{\frac{\delta F}{\delta x}(x,y,z)}{\frac{\delta F}{\delta z}(x,y,z)}$ pokud $\frac{\delta F}{\delta z} \ne 0$

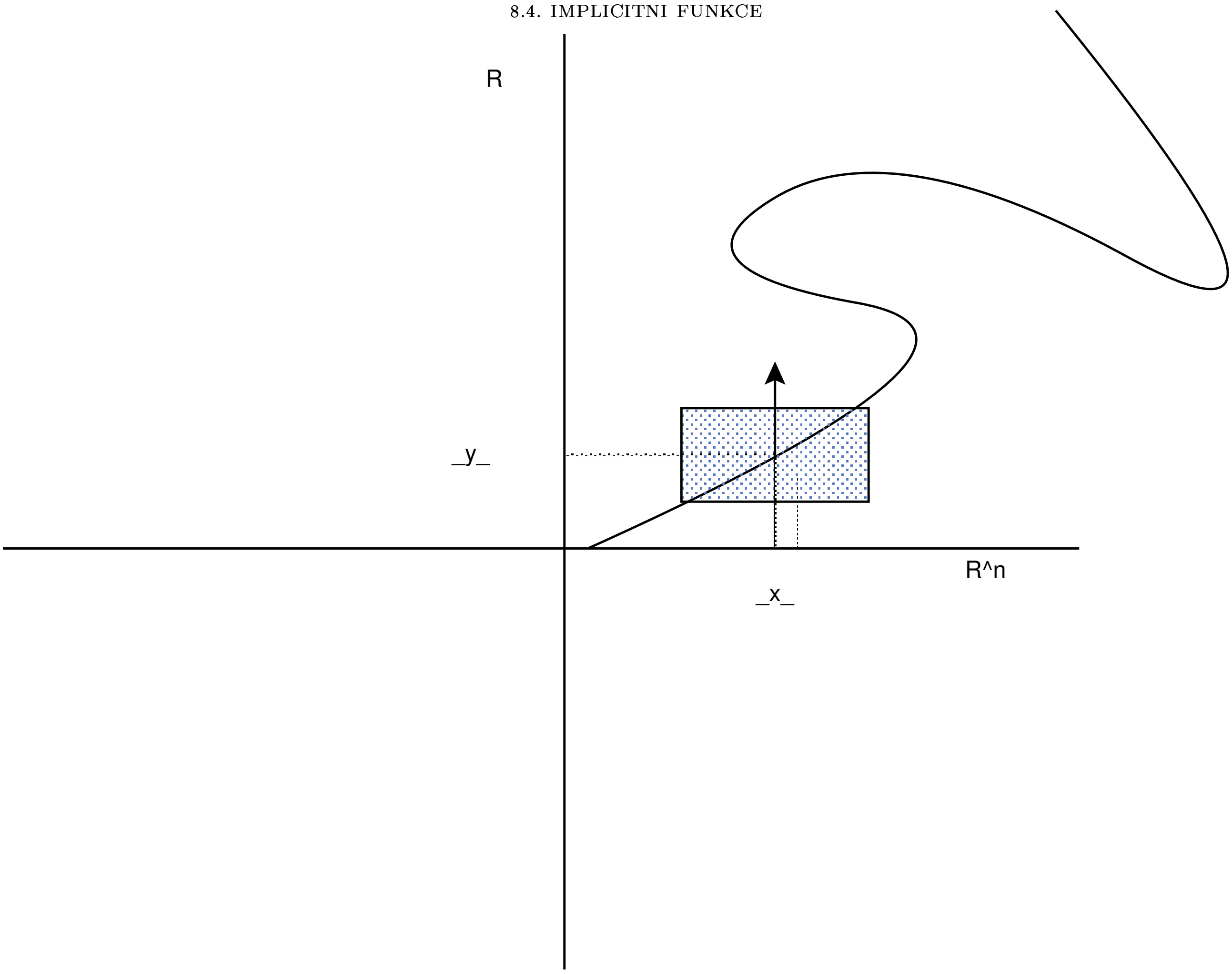
THEOREM. *10. (o implicitni funkci) (lehci verze) TV*
(! - "nejdulezitejsi veta sezony")
Necht $G \subset \mathbf{R}^{\mathbf{n+1}}$ je otevrena.
Necht $\overline{x} \in \mathbf{R}^{\mathbf{n}}, \overline{y} \in \mathbf{R}$ MA32
Necht $F : G \to \mathbf{R}$, ktere spluje:
(i) $F \in \mathcal{C}^1(G)$
(ii) $F(\overline{x}, \overline{y}) = 0$
(iii) $\frac{\delta F}{\delta y}(\overline{x}, \overline{y}) \ne 0$
Potom $\exists$ okoli U bodu $\overline{x}$ a okoli V bodu $\overline{y}$ takove, ze $\forall x \in U \exists! y \in V : F(x, y) = 0$
Piseme-li $y = \varphi(x), x \in U$, pak $\varphi : \mathbf{R}^{\mathbf{n}} \to \mathbf{R}, \varphi \in \mathcal{C}^1(U)$ a plati
$\frac{\delta \varphi}{\delta x_j}(x) = -\frac{\frac{\delta F}{\delta x_j}(x,y)}{\frac{\delta F}{\delta y}(x,y)} \forall x \in U, y \in V$

PROOF. /*nezda se snad ani prilis tezkej ¯pekny myslenky*/

1.krok - dukaz existence implicitni funkce $y = \varphi(x)$, tzv. krabickovani
BUNO $\frac{\delta F}{\delta y}(\overline{x}, \overline{y}) > 0$
$F \in \mathcal{C}^1(G) \Rightarrow \exists \delta_1, \xi \in \mathbf{R}$ takove, ze $\frac{\delta F}{\delta y}(x,y) > 0 \forall x \in B(\overline{x}, \delta_1) \, \forall y \in (y - \xi, y + \xi)$
Potom je funkce $g(t) = F(\overline{x}, t)$ rostouci na intervalu $(y - \xi, y + \xi)$ (derivace podle svisly promenny je kladna)



Tedy $F(\overline{x}, \overline{y} - \xi) < 0, F(\overline{x}, \overline{y} + \xi) > 0$ (nebot $F(\overline{x}, \overline{y} - \xi) < \underbrace{F(\overline{x}, \overline{y})}_{=0} < F(\overline{x}, \overline{y} + \xi)$)
Protoze F je spojita, existuje $\delta_2 > 0, \delta_2 < \delta_1$ tak, ze $F(x, \overline{y} + \xi) > 0 \forall x \in B(\overline{x}, \delta_2)$, $F(x, \overline{y} - \xi) < 0 \forall x \in B(\overline{x}, \delta_2)$/*wow*/



Nyni pro kazde $x \in B(\overline{x}, \delta_2)$ je funkce
$h(t) = F(x,t)$
spojita a rostouci na intervalu $(\overline{y} - \xi, \overline{y} + \xi)$
a navic $h(\overline{y} - \xi) < 0$ a $h(\overline{y} + \xi) > 0$
/*fixnu jeden x - paprsek*/
Tedy ke kazdemu $x \in B(\overline{x}, \delta_2)\, \exists! y \in (\overline{y} - \xi, \overline{y} + \xi)$ takove, ze $F(x,y) = 0$
Vezmeme-li $U = B(\overline{x}, \delta_2), V = (\overline{y} - \xi, \overline{y} + \xi)$, pak mame existenci implicitni funkce
Piseme $y = \varphi(x)$, tedy $F(x, \varphi(x)) = 0 \forall x \in B(\overline{x}, \delta_2)$
2. krok - dukaz spojitosti φ v bode $\overline{x}$ (vzhledem k U)
Zvolme $\varepsilon > 0$. Pak podle jiz dokazaneho (krok 1) $\exists \tau$ tak, ze
$\forall x \in B(\overline{x}, \tau)\, \exists! y \in B(\overline{y}, \varepsilon)$ tak, ze $y = \varphi(x)$
a tedy $\varphi(B(\overline{x}, \tau)) \subset B(\overline{y}, \varepsilon)$, tj. φ je spojita v $\overline{x}$.
3. krok - dukaz diferencialnich vlastnosti: $\varphi \in \mathcal{C}^1$ a tvaru $\frac{\delta \varphi}{\delta x_j}(\overline{x})$
Vime: F ma totalni diferencial (nebot $F \in \mathcal{C}^1(G)$)
$\lim_{k \to o \in \mathbf{R^{n+1}}} \frac{\left|F(x+k) - F(x) - \sum_{j=1}^{n+1} \frac{\delta F}{\delta x_j}(x) k_j\right|}{\|k\|_{\mathbf{R^{n+1}}}} = 0$
tedy specialne /*$n+1$-ni promennou nahradime φ)*/
$\lim_{h \to o \in \mathbf{R^n}} \frac{\left|F(x+h, \varphi(x+h)) - F(x, \varphi(x)) - \sum_{j=1}^{n} \frac{\delta F}{\delta x_j}(x, \varphi(x)) h_j - \frac{\delta F}{\delta y}(x, \varphi(x))(\varphi(x+h) - \varphi(x))\right|}{\|h\|_{\mathbf{R^n}} + |\varphi(x+h) - \varphi(x)|} = 0$
(plyne ze spojitosti φ, tj. $h \to 0 \Rightarrow |\varphi(x+h) - \varphi(x)| \to 0$)
Zvolme ε. Pak $\exists \delta : \forall h : \|h\| < \delta$
$$\left|\underbrace{F(x+h)}_{0} - \underbrace{F(x, \varphi(x))}_{0} - \sum_{j=1}^{n} \frac{\delta F}{\delta x_j}(x, \varphi(x)) h_j - \frac{\delta F}{\delta y}(x, \varphi(x))(\varphi(x+h) - \varphi(x))\right| < \varepsilon \|h\|_{\mathbf{R^n}} + \varepsilon |\varphi(x+h) - \varphi(x)|$$
Dale: $|\varphi(x+h) - \varphi(x)| \le \left|\varphi(x+h) - \sum_{j=1}^{n} \left(-\frac{\frac{\delta F}{\delta x_j}(x,y)}{\frac{\delta F}{\delta y}(x,y)} h_j\right)\right| + \left|\sum_{j=1}^{n} \left(-\frac{\frac{\delta F}{\delta x_j}(x,y)}{\frac{\delta F}{\delta y}(x,y)} h_j\right) - \varphi(x)\right| = I + II$
$II \le c\|h\|$ (c zavisi na $\frac{\delta F}{\delta x}, \frac{\delta F}{\delta y}$)
$I \le \frac{1}{\frac{\delta F}{\delta y}(x,y)} \left|\left(\frac{\delta F}{\delta y}(x,y) \varphi(x+h) - \sum \frac{\delta F}{\delta x_j}(x,y) h_j\right)\right| \le \frac{1}{c_2}(\|h\|_{\mathbf{R^n}} + |\varphi(x+h) - \varphi(x)|)$
Pro $\varepsilon < c_2 : |\varphi(x+h) - \varphi(x)| \le c_3 \|h\|_{\mathbf{R^n}}$
Odtud a z odhadu I mame:
$I \le \frac{\varepsilon}{c_2}(\|h\|_{\mathbf{R^n}} + |\varphi(x+h) - \varphi(x)|) \le \varepsilon \left(\frac{1}{c_2} + c_3\right) \|h\|$

Tedy $D\varphi(x)(h)=\sum_{j=1}^{n}-\frac{\frac{\delta F}{\delta x_j}(x,y)}{\frac{\delta F}{\delta y}(x,y)}h_j$, protoze $\frac{\frac{\delta F}{\delta x_j}}{\frac{\delta F}{\delta y}}$ jsou spojite, mame $\varphi\in\mathcal{C}^1(U)$ □

EXAMPLE 8.4.1. (typicky pisemkovy priklad) Dana mnozina $M=\left\{[x,y]\in\mathbf{R}^2,\left(x^2+y^2\right)^2-2\left(x^2-y^2\right)=0\right\}$ a bod $\left[\frac{\sqrt{3}}{2},\frac{1}{2}\right]$

Dokazte¸ze v jistem okoli bodu $\left[\frac{\sqrt{3}}{2},\frac{1}{2}\right]$ lze popsat mnozinu M jako graf funkce $y=\varphi(x)$ (promenne x) a spocitejte $\varphi\prime\left(\frac{\sqrt{3}}{2}\right)$

Reseni:
overime predpoklady VOIF
def $F(x,y)=\left(x^2+y^2\right)^2-2\left(x^2-y^2\right)$
(i)
potom $F\in\mathcal{C}^\infty\left(\mathbf{R}^2\right)$
(ii) $F\left(\frac{\sqrt{3}}{2},\frac{1}{2}\right)=0$
(iii) $\frac{\delta F}{\delta y}\left(\frac{\sqrt{3}}{2},\frac{1}{2}\right)=4\neq 0$ ($\frac{\delta F}{\delta y}=2\left(x^2+y^2\right)2y+4y$)
Tedy VOIF: $\exists\frac{\sqrt{3}}{2}\in U,\frac{1}{2}\in V\forall x\in U\exists!y\in V,y=\varphi(x),F(x,y)=0$
$\varphi\in\mathcal{C}^1(U)$
$\varphi'\left(\frac{\sqrt{3}}{2}\right)=-\frac{\frac{\delta F}{\delta x}(x,y)}{\frac{\delta F}{\delta y}(x,y)}||_{\left[\frac{\sqrt{3}}{2},\frac{1}{2}\right]}=-\frac{2\left(x^2+y^2\right)2x-4x}{4}||_{\left[\frac{\sqrt{3}}{2},\frac{1}{2}\right]}=0$

NOTE 8.4.2. Zapomeneme-li vzorec $\varphi\prime(x)=-\frac{\frac{\delta F}{\delta x}}{\frac{\delta F}{\delta y}}$, odvodime jej derivovanim podle x rovnice $F(x,\varphi(x))=0$ (chain rule)

8.4.1. sirsi verze. $F(x,y,u,v)=0$
$G(x,y,u,v)=0$
Problemy:
(1) Lze tyto vztahy interpretovat tak, aby definovaly napr. $u(x,y),v(x,y)$ jako funkce promennych x,y?
(2) Lze spocitat $\frac{\delta u}{\delta x},\frac{\delta u}{\delta y},\frac{\delta v}{\delta x},\frac{\delta v}{\delta y},Du,Dv$?

THEOREM. *11. (o implicitni funkci vice promennych) (tezsi verze) (BEZ DUKAZU)*
Necht $G\subset\mathbf{R}^{n+m}$ je otevrena.
$F_j:\mathbf{R^{n+m}}\to\mathbf{R},j=1,\ldots,m$
$\overline{x}\in\mathbf{R}^n,\overline{y}\in\mathbf{R}^m$
Necht
(i) $F_j\in\mathcal{C}^1(G)\,\forall j=1,\ldots,m$
(ii) $F_j(\overline{x},\overline{y})=0\forall j=1,\ldots,m$
(iii) $\left|\frac{\delta(F_1,\ldots,F_m)}{\delta(y_1,\ldots,y_m)}\right|_{(\overline{x},\overline{y})}\neq 0$, *kde* $\left|\frac{\delta(F_1,\ldots,F_m)}{\delta(y_1,\ldots,y_m)}\right|=\begin{vmatrix}\frac{\delta F_1}{\delta y_1}&\cdots&\frac{\delta F_1}{\delta y_m}\\ \vdots&\ddots&\vdots\\ \frac{\delta F_m}{\delta y_1}&\cdots&\frac{\delta F_m}{\delta y_m}\end{vmatrix}$
Potom
$\exists$ okoli $U\subset\mathbf{R^n}$ bodu $\overline{x}$ a okoli $V\subset\mathbf{R}^m$ bodu $\overline{y}$ takove, ze
$\forall x\in U\exists!y\in V:F_j(x,y)=0\forall j=1,\ldots,m$
piseme-li $y_j=\varphi_j(x)$, pak $\varphi_j\in\mathcal{C}^1(U),j=1,\ldots,m$ a plati
$\frac{\delta\varphi_j}{\delta x_k}(x)=-\frac{\left|\frac{\delta(F_1,\ldots,F_m)}{\delta(y_1,\ldots,y_{j-1},x_k,y_{j+1},\ldots,y_m)}\right|}{\left|\frac{\delta(F_1,\ldots,F_m)}{\delta(y_1,\ldots,y_m)}\right|}$ *(v citateli na miste "y_j" piseme "x_k")*
(dukaz - indukci podle m)

NOTE 8.4.3. $\frac{\delta F_j}{\delta x_k}$ lze take spocitat derivovanim rovnic $F_j\left(x_1,\ldots,x_n,\varphi_1(x_1,\ldots x_n),\ldots,\varphi_m(x_1,\ldots x_n)\right)=0$ podle promenne x_k (chain rule) $\to$ soustava linearnich rovnic pro $\frac{\delta\varphi_j}{\delta x_k}$

EXAMPLE 8.4.4. Jsou dany vztahy
$xe^y+uz-\cos v=2$
$u\cos y+x^2v-yz^2=1$
a bod $[2,0,1,1,0]$
Dokazte, ze tyto vztahy definuji a okoli bodu $[2,0,1,1,0]$ funkce $u(x,y,z),v(x,y,z)$
Spocitejte $Du(2,0,1)$ a $Dv(2,0,1)$, $\exists$-li
$n=3,m=2$
def: $F(x,y,z,u,v)=xe^y+uz-\cos v-2$
$G(x,y,z,u,v)=u\cos y+x^2v-yz^2-1$
(i) $F,G\in\mathcal{C}^\infty$
$F(2,0,1,1,0)=0$
$G(2,0,1,1,0)=0$
$\left|\frac{\delta(F,G)}{\delta(u,v)}\right|_{[2,0,1,1,0]}=\begin{vmatrix}\frac{\delta F}{\delta u}&\frac{\delta F}{\delta v}\\ \frac{\delta G}{\delta u}&\frac{\delta G}{\delta v}\end{vmatrix}_{[2,0,1,1,0]}=\begin{vmatrix}z&\sin v\\ \cos y&x^2\end{vmatrix}_{[2,0,1,1,0]}=4\neq 0$

V11$\Rightarrow \exists$ na nejakem okoli funkce $u(x,y,z), v(x,y,z) \in \mathcal{C}^1, u(2,0,1) = 1, v(2,0,1) = 0$

$\frac{\delta u}{\delta x}(x,y,z) = -\frac{\frac{\delta(F,G)}{\delta(x,v)}}{\frac{\delta(F,G)}{\delta(u,v)}} = -\frac{1}{4}\begin{vmatrix} \frac{\delta F}{\delta x} & \frac{\delta F}{\delta v} \\ \frac{\delta G}{\delta x} & \frac{\delta G}{\delta v} \end{vmatrix}_{[2,0,1,1,0]} = -\frac{1}{4}\begin{vmatrix} e^y & 0 \\ 2xv & 4 \end{vmatrix} = -e^y = -1$

$\frac{\delta u}{\delta y}(x,y,z) = -\frac{\frac{\delta(F,G)}{\delta(y,v)}}{\frac{\delta(F,G)}{\delta(u,v)}} = -\frac{1}{4}\begin{vmatrix} \frac{\delta F}{\delta y} & \frac{\delta F}{\delta v} \\ \frac{\delta G}{\delta y} & \frac{\delta G}{\delta v} \end{vmatrix}_{[2,0,1,1,0]} = -\frac{1}{4}\begin{vmatrix} xe^y & 0 \\ TODO & 4 \end{vmatrix} = -2$

$\frac{\delta u}{\delta z}(x,y,z) = -\frac{1}{4}\begin{vmatrix} \frac{\delta F}{\delta z} & 0 \\ \frac{\delta G}{\delta z} & 4 \end{vmatrix}_{[2,0,1,1,0]} = -\frac{1}{4}\begin{vmatrix} u & 0 \\ 2xy & 4 \end{vmatrix} = -1$

Tedy $\nabla u(2,0,1) = (-1,-2,-1)$

Diferencial zadarmo existuje, nebot $F_j \in \mathcal{C}^1$

$Du(2,0,1)(h_1,h_2,h_3) = -h_1 - 2h_2 - h_3$

Dv analogicky.

8.5. extremy funkci vice promennych

Definition 8.5.1. Necht f je funkce n promennych, necht $M \subset D(f)$. Rekneme, ze f ma v bode $a \in M$ **minimum** vzhledem k mnozine M, jestlize $\forall x \in M : f(x) \geq f(a)$

(klicovy pojem) Rekneme, ze f ma v bode $a \in M$ **lokalni minimum** vzhledem k mnozine M, jestlize $\exists r > 0 : \forall x \in M \cap B(a,r) : f(x) \geq f(a)$

Analogicky definujeme **maximum, lokalni maximum a ostrost**

Theorem. *12. (vzah extremu k parcialnim derivacim) LV*
(nutna podminka $\exists$extremu)
Necht $G \subset \mathbf{R^n}$ je otevrena, necht $a \in G$, necht $f : G \to \mathbf{R}$, necht f ma v a lokalni extrem.
Necht $\exists$ vsechny parcialni derivace prvniho radu v bode a.
Pak $\frac{\delta f}{\delta x_1}(a) = \ldots = \frac{\delta f}{\delta x_n}(a) = 0$
(dukaz obdobne jako u funkci jedne promenne)

-

Theorem. *13. (Lagrangeova o vazanych extremech) (BEZ DUKAZU)*
$G \subset \mathbf{R^r}$
$f, g_1, \ldots, g_s \in \mathcal{C}^1(G), r, s \in \mathbf{N}$
Necht $M = \{x \in G; g_j(x) = 0, j = 1, \ldots, s\}$ (g_j - vazby)
Necht $a \in M$
Pokud f ma v a lokalni extrem vzhledem k M a navic vektory $\nabla g_1(a), \ldots, \nabla g_s(a)$ jsou linearne nezavisle, potom existuji jednoznacne urcena realna cisla $\lambda_1, \ldots, \lambda_s$ takova, ze
$Df(a) + \sum_{j=1}^{s} Dg_j(a)\lambda_j = \emptyset$ (nulove zobrazeni)
Tedy

$$r \text{ rovnic} \begin{cases} \frac{\delta f}{\delta x_1}(a) + \frac{\delta g_1}{\delta x_1}(a)\lambda_1 + \ldots + \frac{\delta g_s}{\delta x_1}(a)\lambda_s = 0 \\ \vdots \\ \frac{\delta f}{\delta x_r}(a) + \frac{\delta g_1}{\delta x_r}(a)\lambda_1 + \ldots + \frac{\delta g_s}{\delta x_r}(a)\lambda_s = 0 \end{cases}$$

Example 8.5.2. obecne: veta nam dava r rovnic pro nezname: $\lambda_1, \ldots, \lambda_s, x_{s+1}, \ldots, x_r$ (zbyvajici promenne $x_1, \ldots, x_s$ dostaneme jako funkce promennych $x_{s+1}, \ldots, x_r$)

$\Rightarrow$ spocitame $x_{s+1}, \ldots, x_r$, dopocteme z VOIF $x_1, \ldots, x_s$ a mame kandidaty na extrem.

.

Example 8.5.3. $f(x,y) = x + y, M = \left\{[x,y] \in \mathbf{R}^2; x, y \geq 0, g(x,y) = x^3 + y^3 - 2xy = 0\right\}$

Najdete extremy f na M

$\frac{\delta f}{\delta x} = 1$, $\frac{\delta f}{\delta y} = 1$, $\frac{\delta g}{\delta x} = 3x^2 - 2y$, $\frac{\delta g}{\delta y} = 3y^2 - 2x$

$\Rightarrow \exists\lambda$

$1 + \lambda\left(3x^2 - 2y\right) = 0$

$1 + \lambda\left(3y^2 - 2x\right) = 0$

$x^3 + y^3 - 2xy = 0$

->mame 3 rovnice pro nezname λ, x, y

(1)-(2)$\Rightarrow \lambda\left(3\left(x^2 - y^2\right) + 2(x - y)\right) = 0$

$\lambda(x - y)(3(x + y) + 2) = 0$

(1)$\Rightarrow \lambda \neq 0 \Rightarrow$ bud $y = x$

nebo $x + y = -\frac{2}{3}$ - nelze, nebot $x, y \geq 0$

dosad $y = x$ do (3): $2x^3 - 2x^2 = 0 \Rightarrow \begin{matrix} x = 0 \\ x = 1 \end{matrix}$

dopocitame y : $\begin{matrix} x = 0 \Rightarrow y = 0 \\ x = 1 \Rightarrow y = 1 \end{matrix}$

kandidati na extrem: $[0,0], [1,1]$

$f(0,0) = 0 \to min$

$f(1,1) = 2 \to max$

NOTE 8.5.4. k prikladu MA37
vrstevnice f: $f(x,y)=c$
V13 dava kandidaty na extrem
postup: dano f,g (prip. $g_1,\dots,g_n$)
Def $G(x,y,\lambda)=f(x,y)+\lambda g(x,y)$
Pak V13 tvrdi, ze ma-li f v bode a vazany extrem, pak $\exists!\lambda_0$ takove, ze $G(x,y,\lambda)$ ma v bode (a_1,a_2,λ_0) extrem.
Tedy $\nabla G(a_1,a_2,\lambda_0)=o$
$\frac{\delta G}{\delta x}=\frac{\delta f}{\delta x}+\lambda_0\frac{\delta g}{\delta x}=0$
$\frac{\delta G}{\delta y}=\frac{\delta f}{\delta y}+\lambda_0\frac{\delta g}{\delta y}=0$
$\frac{\delta G}{\delta \lambda}=g(x,y)=0$
-> 3 rovnice pro nezname x,y,λ

THEOREM. *13'. (Lagrangeova o multiplikatoru pro jednu vazbu) TV*
$G\subset\mathbf{R^{n+1}}, f,g\in\mathcal{C}^1(G)$
$a\in\mathbf{R^{n+1}},\nabla g(a)\neq o$ *(tedy chceme, aby byly linearne zavisle)*
Necht f ma v bode a lokalni extrem vzhledem k mnozine $M=\{x\in G, g(x)=0\}$
Potom $\exists!\lambda\in\mathbf{R}$ tak, ze $Df(a)+\lambda Dg(a)=o$
(tj. $\nabla f(a)+\lambda\nabla g(a)=o$)
*/*volny extrem, pak zderivovat vazbu, podle implicitni dostanem λ a dopocitame, ze plati i pro ostatni promenne*/*

PROOF. Prevedeme vazany extrem na volny extrem
$\nabla g(a)\neq o\Rightarrow\exists j\frac{\delta g}{\delta x_j}\neq 0$, BUNO necht $\frac{\delta g}{\delta x_{n+1}}\neq 0$
Tim padem g splnuje predpoklady VOIF.
Tedy $\exists$ okoli $U\subset\mathbf{R^n}$ bodu $\alpha=[a_1,\dots,a_n]$ a $\exists$ okoli $V\subset\mathbf{R}$ bodu a_{n+1} tak, ze
$\forall[x_1,\dots,x_n]\,\exists!x_{n+1}\in V: g(x_1,\dots,x_n,x_{n+1})=0$
znaceni $x_{n+1}=\varphi(x_1,\dots,x_n)$ (vime: $\varphi\in\mathcal{C}^1(U)$)
$F(x_1,\dots,x_n)=f(_1,\dots,x_n,\varphi(x_1,\dots,x_n))$
Potom $F\in\mathcal{C}^1(U)$ a ma v bode a extrem $\Rightarrow DF(\alpha)=0$, tj. $\frac{\delta F}{\delta x_j}(\alpha)=0\forall j=1,\dots,n$
Jak ziskat λ? Podle implicitni promenne. Vime: $\exists!\lambda\in\mathbf{R}$ tak, aby $\frac{\delta f}{\delta x_{n+1}}(a)+\lambda\underbrace{\frac{\delta g}{\delta x_{n+1}}}_{\neq 0}(a)=0$ (linearni rovnice o jedne nezname)
$j=1,\dots,n: 0=\frac{\delta F}{\delta x_j}(\alpha)=\frac{\delta}{\delta x_j}(f(x_1,\dots,x_n,\varphi(x_1,\dots,x_n)))=\frac{\delta f}{\delta x_j}(a)+\frac{\delta f}{\delta x_{n+1}}(a)\frac{\delta\varphi}{\delta x_j}(\alpha)$ (1) (chain rule)
Zderivujeme $g(x_1,\dots,x_n,\varphi(x_1,\dots,x_n))=0$ (derivujeme vazbu)
$\forall j=1,\dots,n:\frac{\delta g}{\delta x_j}(a)+\frac{\delta g}{\delta x_{n+1}}(a)\frac{\delta\varphi}{\delta x_j}(\alpha)=0$ (2)
$(1)+\lambda(2)=\frac{\delta f}{\delta x_j}(a)+\frac{\delta f}{\delta x_{n+1}}(a)\frac{\delta\varphi}{\delta x_j}(\alpha)+\lambda\left(\frac{\delta g}{\delta x_j}(a)+\frac{\delta g}{\delta x_{n+1}}(a)\frac{\delta\varphi}{\delta x_j}(\alpha)\right)=0$
To jest $\frac{\delta f}{\delta x_j}(a)+\lambda\frac{\delta g}{\delta x_j}(a)+\frac{\delta\varphi}{\delta x_j}(\alpha)\underbrace{\left(\frac{\delta f}{\delta x_{n+1}}(a)+\lambda\frac{\delta g}{\delta x_{n+1}}(a)\right)}_{=0}=0$
/*myslenka: λ dostaneme podle vyjimecne promenne, ale pak musime dokazat, ze to plati i pro vsechny ostatni*/
$=\frac{\delta f}{\delta x_j}(a)+\lambda\frac{\delta g}{\delta x_j}(a)=0\forall j=1,\dots,n$
tedy $\forall j=1,\dots,n+1$ □

8.5.1. /*Definice z LA*/.

DEFINITION 8.5.5. Necht $(a_{ij})_{i,j=1}^n$ je symetricka (ctvercova) matice. Pak zobrazeni $B:\mathbf{R^n}\times\mathbf{R^n}\to\mathbf{R}$ def. predpisem $B(h,k)=\sum_{i,j=1}^n a_{ij}h_ik_j$ je bilinearni forma
a zobrazeni $Q:\mathbf{R^n}\to\mathbf{R}$ definovane predpisem $Q(h)=B(h,h)$ je kvadraticka forma.
(poznamka: $Q(\lambda h)=\lambda^2Q(h)$)
Rekneme, ze B (nebo Q) je pozitivne definitni nebo negativni definitni a nebo indefinitni, jestlize:
$B(h,h)>0\forall h\neq o$ nebo $B(h,h)<0\forall h\neq o$ a nebo $\exists h_1,h_2 B(h_1,h_1)<0<B(h_2,h_2)$
$(A\in M(u\times v), B(u,v)=u^tAv)$

EXAMPLE 8.5.6. skalarni soucin je pozitivne definitni

/*konec definic z LA*/
Add “vety” pro urcovani PD, ...

LEMMA. $B:\mathbf{R^n}\times\mathbf{R^n}\to\mathbf{R}$ *bilinearni forma je pozitivne definitni prave kdyz* $\exists\varepsilon>0\forall h: B(h,h)\geq\varepsilon\|h\|^2$

PROOF. “$\Leftarrow$” trivialni
“$\Rightarrow$”
projekce na jednotkovou sferu v $\mathbf{R^n}$ $S=\{h\in\mathbf{R^n},\|h\|=1\}$
S je kompaktni.
$\left\|\frac{h}{\|h\|}\right\|=\frac{\|h\|}{\|h\|}=1\Rightarrow\frac{h}{\|h\|}\in S$
$Q(h)=B(h,h)$ je spojita funkce na $\mathbf{R^n}$. Tedy nabyva sveho minima na S.

Tedy $\exists h_0, \|h_0\| = 1 : Q(h) \geq Q(h_0) \forall h \in S$. Oznacme $\varepsilon = Q(h_0) > 0$, nebot B je pozitivne definitni
Potom pro libovolne $h \in \mathbf{R^n} \neq 0 : Q(h) = Q\left(\|h\| \frac{h}{\|h\|}\right) = \|h\|^2 Q\left(\frac{h}{\|h\|}\right) \geq \|h\|^2 Q(h_0) = \varepsilon \|h\|^2$ □

8.5.2. postacujici podminka.

THEOREM. *14. (postacujici podminky pro existenci lokalniho extremu) TV*
Necht $G \subset \mathbf{R}^2$ je otevrena, $F \in \mathcal{C}^2(G), a \in G$
Necht $Df(a) = o$
(1) Je-li $D^2f(a)$ pozitivne definitni, pak f ma v a lokalni minimum.
(2) Je-li $D^2f(a)$ negativne definitni, pak f ma v a lokalni maximum.
(3) Je-li $D^2f(a)$ indefinitni, pak f nema v a lokalni extrem.

PROOF. **(1)**
/*docela neprijemny:
1. - $D^2f(x)$ PD na okoli a (koule) - odhady, spojitost D^2
2. dok., ze na tom okoli $\forall x \in B(a,r) \exists \xi \in (0,1) : f(x) - f(a) - Df(a)(x-a) = \frac{1}{2}D^2f(a+\xi(x-a))(x-a)(x-a)$ - subst na funkci 1 promenne (jako Lagr. VOStr. Hodnote) + Lagrangeuv tvar zbytku
3. pouzit -OK
*/
1. krok $D^2f(a)$ pozitivne def. $\Rightarrow D^2f(x)$ je take poz. def. $\forall x$ z nejakeho okoli
$D^2f(x)(h,h) = D^2f(x)(h,h) + D^2f(a)(h,h) - D^2f(a)(h,h)$
$\geq \varepsilon\|h\|^2 + \sum_{i,j=1}^n \left(\frac{\delta^2 f}{\delta x_i \delta x_j}(x) - \frac{\delta^2 f}{\delta x_i \delta x_j}(a)\right) h_i h_j$
$\geq \varepsilon\|h\|^2 - \sum_{i,j=1}^n \left|\frac{\delta^2 f}{\delta x_i \delta x_j}(x) - \frac{\delta^2 f}{\delta x_i \delta x_j}(a)\right| |h_i h_j|$
$\geq \varepsilon\|h\|^2 - \sum_{i,j=1}^n \left|\frac{\delta^2 f}{\delta x_i \delta x_j}(x) - \frac{\delta^2 f}{\delta x_i \delta x_j}(a)\right| \|h_i h_j\|$
Protoze $f \in \mathcal{C}^2 : \exists r > 0 \forall x \in B(a,r) : \left|\frac{\delta^2 f}{\delta x_i \delta x_j}(x) - \frac{\delta^2 f}{\delta x_i \delta x_j}(a)\right| < \frac{\varepsilon}{2n^2} \forall i,j$
Pro takove x mame:
$D^2f(x)(h,h) \geq \varepsilon\|h\|^2 - \sum_{i,j=1}^n \frac{\varepsilon}{2n^2}\|h\|^2 = \varepsilon\|h\|^2 - \frac{\varepsilon}{2}\|h\|^2 = \frac{\varepsilon}{2}\|h\|^2$
2. krok $\forall x \in B(a,r) \exists \xi \in (0,1) : f(x) - f(a) - Df(a)(x-a) = \frac{1}{2}D^2f(a+\xi(x-a))(x-a)(x-a)$
Def $F(t) = f(a + t(x-a))$ /*like Vo stredni hodnote*/
pak $F \in \mathcal{C}^2([0,1]), F(0) = f(a), F(1) = f(x)$
Lagr. tvar zbytku:
$F(1) - F(0) - F\prime(0)(1-0) = \frac{1}{2}F\prime\prime(\xi)(1-0)^2 \ (\exists \xi \in (0,1))$
$f(x) - f(a) - F\prime(0) = \frac{1}{2}F\prime\prime(\xi)$
Mame: $F\prime(t) = \sum_{j=1}^n \frac{\delta f}{\delta x_j}(a + t(x-a))(x_j - a_j)$, tedy $F\prime(0) = \sum_{j=1}^n \frac{\delta f}{\delta x_j}(a)(x_j - a_j) = Df(a)(x-a)$
$F\prime\prime(t) = \sum_{j=1}^n (x_j - a_j)\frac{d}{dt}\left(\frac{\delta f}{\delta x_j}(a + t(x-a))\right) = \sum_{j=1}^n (x_j - a_j)\sum_{i=1}^n \frac{\delta^2 f}{\delta x_i \delta x_j}(a + t(x-a))(x_j - a_j)$
$= D^2f(a + t(x-a))(x-a, x-a)$
(Dosad $t = \xi \Rightarrow$ 2. krok)
3. krok - tvrzeni
$\forall x \in B(a,r)$
$f(x) - f(a) - \underbrace{Df(a)(x-a)}_{0} = \frac{1}{2}\underbrace{D^2f(a+\xi(x-a))(x-a, x-a)}_{>0} \Rightarrow f(x) > f(a) \Rightarrow f$ ma v a lokalni minimum
(2) analogicky $-f$ misto f
(3) /*docela lehky - vzit ty 2 smery, subst na jeden smer, $\prime,\prime\prime$ a z toho, ze jedno je min a druhy max*/
indefinitni $\Rightarrow \exists u, v \in \mathbf{R^n}, u \neq o, v \neq o : D^2f(a)(u,u) > 0 \& D^2f(a)(v,v) < 0$
def $g_1, g_2 : \mathbf{R} \to \mathbf{R}$
$g_1(t) = f(a + tu), t \in$ okoli 0
$g_2(t) = f(a + tv), t \in$ okoli 0
/*dokazeme, ze k jedne je max a k druhe min*/
$g_1\prime(0) = \frac{d}{dt}(f(a_1 + tu_1, \ldots, a_n + tu_n))|_{t=0} = \sum_{i=1}^n \frac{\delta f}{\delta x_i}(a + tu)u_i = \sum_{i=1}^n \frac{\delta f}{\delta x_i}(a)u_i = Df(a)(u) = 0$
$$g_1\prime\prime(0) = \sum_{i=1}^n \underbrace{\left(\sum_{j=1}^n \frac{\delta f}{\delta x_i x_j}(a + tu)u_j\right)}_{=\left(\frac{\delta f}{\delta x_i}(a+tu)\right)\prime} u_i|_{t=0} = D^2f(a)(u,u) > 0$$
$\Rightarrow g_1$ ma v 0 lokalni minimum
podobne dostaneme, ze $g_2\prime(0) = 0, g_2\prime\prime(0) = D^2f(a)(v,v) < 0$, g_2 ma v 0 lokalni maximum
$\Longrightarrow f$ nema v a lokalni extrem □

EXAMPLE 8.5.7. Naleznete lokalni extremy funkce f na $\mathbf{R}^2$, jestlize $f(x,y) = x^4 + y^4 - x^2 - 2xy - y^2$
$\frac{\delta f}{\delta x}(x,y) = 4x^3 - 2x - 2y = 0$
$\frac{\delta f}{\delta y}(x,y) = 4y^3 - 2x - 2y = 0$
$x^3 - y^3 = 0 \Rightarrow x = y$
$4x^3 - 4x = 0$

$x\left(x^2-1\right)=x(x+1)(x-1)=0$
kandidati $[0,0],[1,1],[-1,-1]$
$\frac{\delta^2 f}{\delta x^2}(x,y)=12x^2$
$\frac{\delta^2 f}{\delta xy}(x,y)=-2$
$\frac{\delta^2 f}{\delta y^2}(x,y)=12y^2-2$
$\begin{pmatrix} 12x^2-2 & -2 \\ -2 & 12y^2-2 \end{pmatrix}$
$[1,1]:\begin{pmatrix} 10 & -2 \\ -2 & 10 \end{pmatrix}$ je PD - maximum
$[0,0]:\begin{pmatrix} -2 & -2 \\ -2 & -2 \end{pmatrix}$ neni PD, ND, je NSD -> vime prd.
$f(x,x)=2x^4-4x^2=2x^2\left(x^2-2\right)$ - v 0 je lokalni maximum.
$f(x,-x)=2x^4$ - v 0 je lokalni minimum
$\Rightarrow$ v $[0,0]$ neni extrem

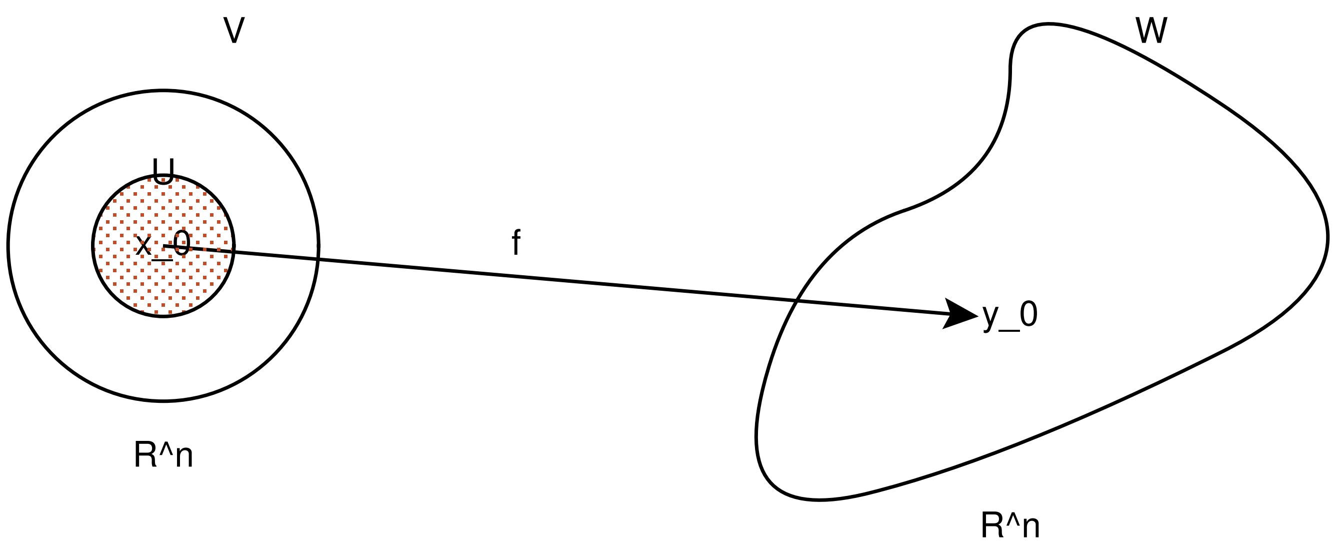
8.6. regularni zobrazeni

DEFINITION 8.6.1. Necht f je zobrazeni n promennych s hodnotami v $\mathbf{R^n}$. Zobrazeni f nazveme **difeomorfismem** na otevrene mnozine $U \subset \mathbf{R^n}$, jestlize
(i) f je proste na U
(ii) $f \in \mathcal{C}^1(W)$
(iii) $W=f(u)$ je otevrena podmozina prostoru $\mathbf{R^n}$
(iv) $f^{-1} \in \mathcal{C}^1(f(W))$

_[3]

DEFINITION 8.6.2. Zobrazeni f definovane na neprazdne mnozine $G \subset \mathbf{R^n}$ s hodnotami v $\mathbf{R^n}$ nazyvame **regularnim zobrazenim** na otevrene mnozine G, jeslize plati:
(i) f je proste zobrazeni na mnozine G
(ii) $f \in \mathcal{C}^1(G)$
(iii) **Jacobián zobrazení** f (tj. determinant matice reprezentujici totalni diferencial) je nenulovy v kazdem bode mnoziny G /*∇f je matice protoze zobrazeni je $\mathbf{R^n} \to \mathbf{R^n}$*/

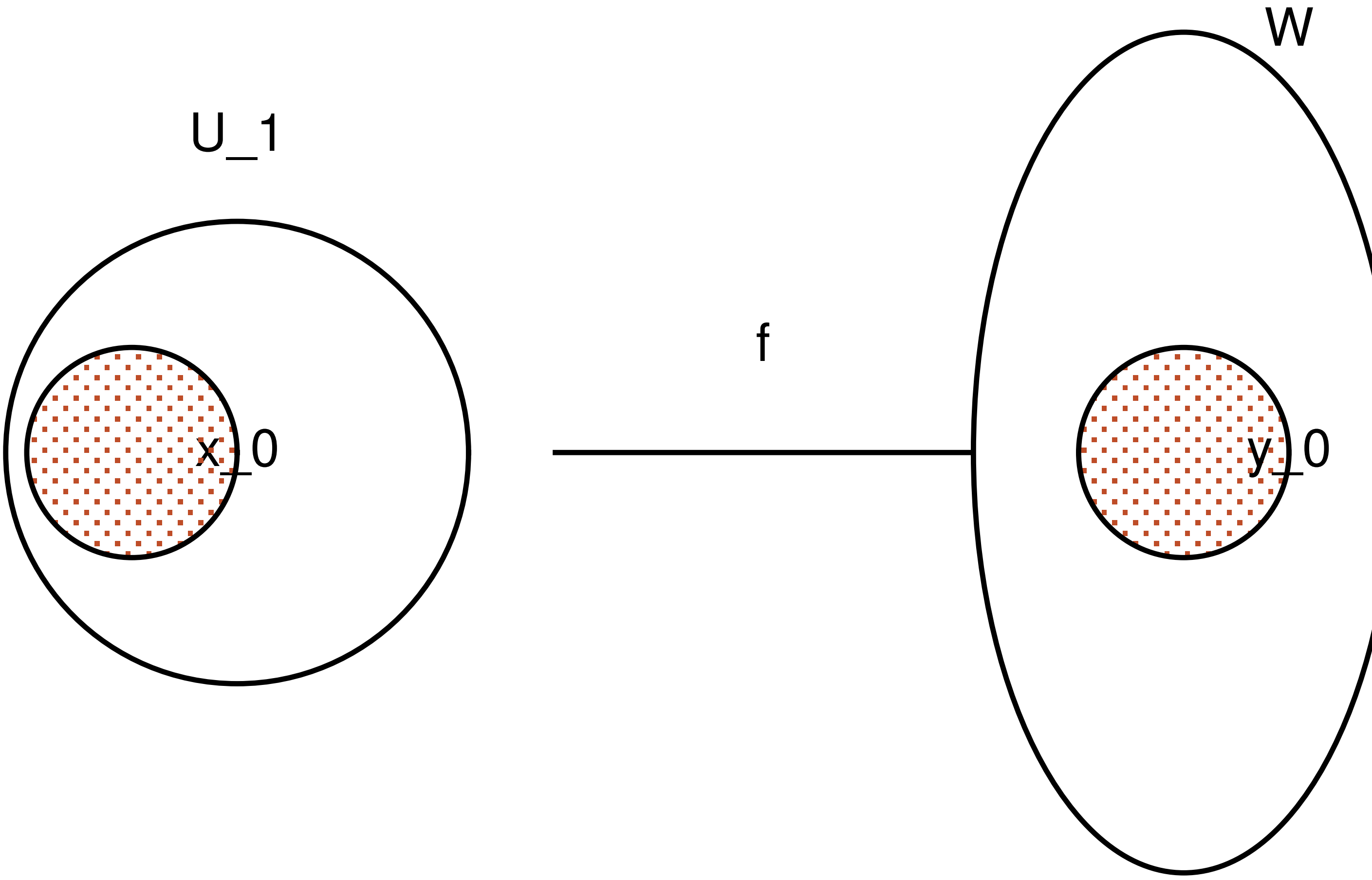
THEOREM. *15. (lokalni difeomorfismus) TV*
Necht f je zobrazeni n promennych, $f: V \to \mathbf{R^n}$, ktere je tridy $\mathcal{C}^1$ na jistem okoli $U \subset V$ bodu $x_0 \in \mathbf{R^n}$ a $Df(x_0)$ je regularni (Jacobian f je nenulovy). Pak $\exists$ okoli $U \subset V$ bodu x_0 tak, ze zobrazeni f zuzene na U je difeomorfismus.



PROOF. /*VOIF^{-1}, machinace s otevrenosti*/
def $F: \mathbf{R^n} \times V \to \mathbf{R^n}$
$F(y,x)=f(x)-y$
$F(y,x)=o$
$y_0=f(x_0)$
chci nalezt $x=f^{-1}(y)$ $(x=x(y))$
pouzijeme VOIF

[3]feo = osklivy (špan.)

predpoklady:
(i) $F(x,y) = (F_1(x,y), \ldots, F_n(x,y))$
$F_j(x,y) = f_j(x) - y_j$
$F_j \in \mathcal{C}^1(\mathbf{R^n} \times V)$
(ii) $F(y_0, x_0) = o$
(iii) $\frac{\delta(F_1,\ldots,F_n)}{\delta(x_1,\ldots,x_n)}|_{(y_0,x_0)} = \begin{pmatrix} \frac{\delta f_1}{\delta x_1} & \frac{\delta f_n}{\delta x_1} \\ \frac{\delta f_1}{\delta x_n} & \frac{\delta f_n}{\delta x_n} \end{pmatrix} |_{(y_0,x_0)} = Df(x_0)$ - regularni $\Rightarrow \left|\frac{\delta(F_1,\ldots,F_n)}{\delta(x_1,\ldots,x_n)}\right| \neq 0$
pouzijeme vetu.



$\exists U_1, W : \forall y \in W \exists! x \in U_1 : f(x) = y (F(x,y) = 0)$ (U_1 okoli x_0, W okoli y_0)
$/{*}x = \varphi(y), \varphi \in \mathcal{C}^1(W), f(x) = y \Rightarrow \varphi = f^{-1}$, a tedy $x = f^{-1}(y), f^{-1} \in \mathcal{C}^1(W){*}/$
Polozime: $U = U_1 \cap f^{-1}(W)$
$x_0 \in U_1$ (VOIF)
$x_0 \in f^{-1}(W)$ nebot $f(x_0) \in W$
$\Rightarrow x_0 \in U$
je U otevrena? U_1 je ot (VOIF)
$f^{-1}(W)$ je ot. (spojity vzor otevrene mnoziny) - f spoj.
$f\left(U_1 \cap f^{-1}(W)\right)$ je ot. protoze je to vzor (ve smyslu zobr. f^{-1}) otevrene mnoziny U
je spojite, protoze $f^{-1} \in \mathcal{C}^1(U_1)$ (impl. funkce) □

Theorem. *16. (o regularnim zobrazeni)*
Necht f je regularni zobrazeni na mnozine $G \neq \emptyset \subset \mathbf{R^n}$. Potom f^{-1} je regularni zobrazeni na $f(G)$.

Proof. Kombinujeme V15 s V o diff. sl. zobr.
$x \in G \Rightarrow f(x) \in f(G)$
$y \in f(G) \Rightarrow \exists x \in G : f(x) = y$
Vime: f je regularni. $Df(x)$ je regularni $\forall x \in G$ (a $f \in \mathcal{C}^1(G)$)
$\overset{V15}{\Rightarrow}$ f je lokalnim difeomorfismem ve vsech bodech mnoziny G
Musime dokazat:
(i) $f(G)$ je otevrena
(ii) f^{-1} je proste na $f(G)$
(iii) $f^{-1} \in \mathcal{C}^1(f(G))$
(iv) $\left|D\left(f^{-1}\right)(x)\right| \neq 0 \forall x \in f(G)$
(i)-(iii) plynou okamzite z V15 (iii-i-ii)
(iv): $f \bullet f^{-1} = id$
$f, f^{-1}, id \in \mathcal{C}^1$VODSZ$\Rightarrow Df(x) * D\left(f^{-1}\right)(x) = I$

$Df(x)$ je regularni $\Rightarrow Df^{-1}(x)$ je regularni matice $\Rightarrow$ (iv) $\square$

CHAPTER 9

Konvergence posloupnonsti a rad funkci

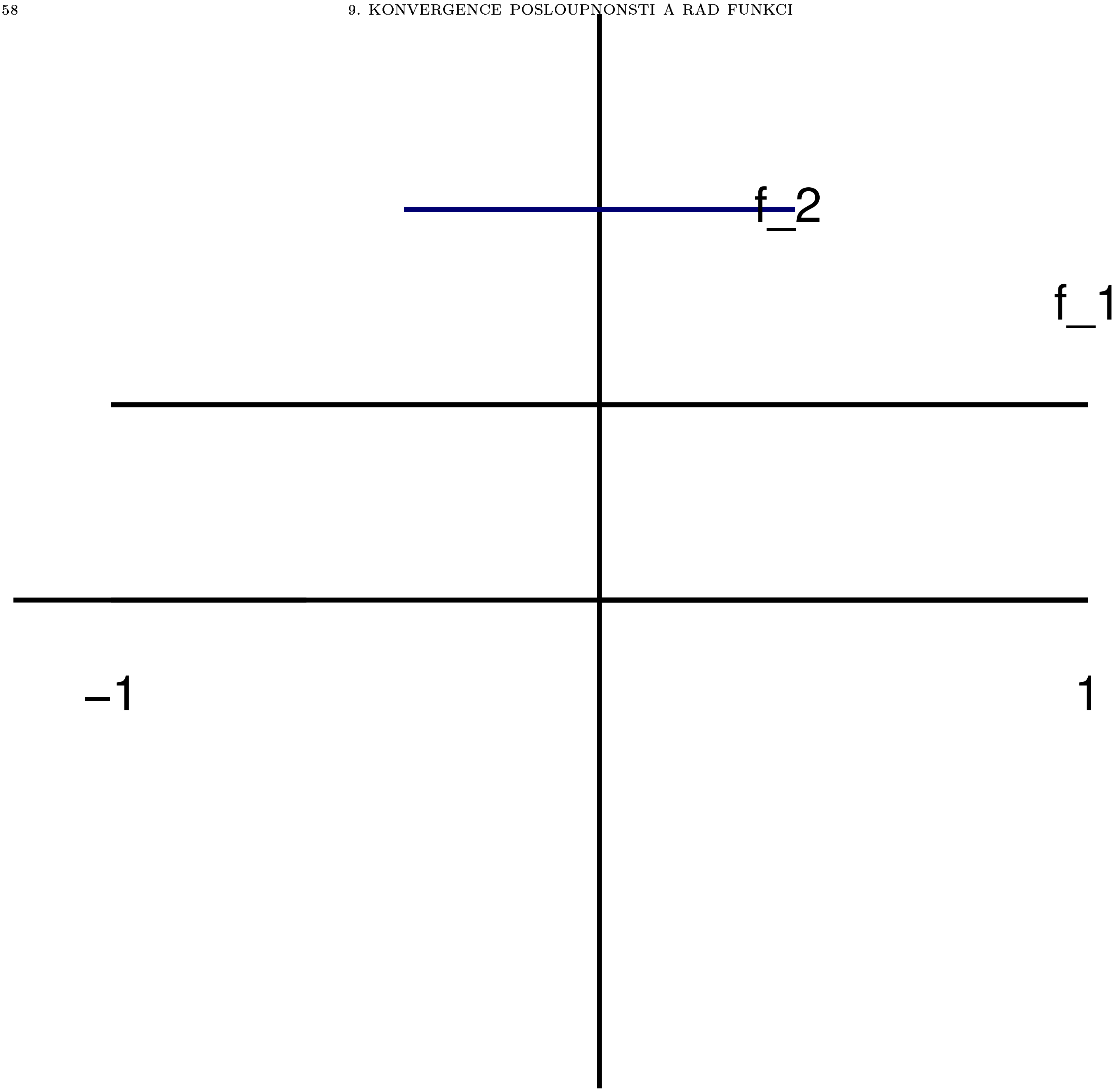
9.1. bodova a stejnomerna konvergence posloupnosti funkci

DEFINITION 9.1.1. Necht M je podmozina metrickeho prostoru (P,ρ). Necht $\{f_n\}_{n=1}^{\infty}$ je posloupnost funkci definovanych na M s hodnotami v $\mathbf{R}$

(klicovy pojem) Rekneme, ze f_n **konverguji bodove** k funkci $f : M \to \mathbf{R}$, jeslize $\forall x \in M \forall \varepsilon \exists n_0 : \forall n \geq n_0 : |f_n(x) - f(x)| < \varepsilon$

NOTE 9.1.2. 1) Bodovou konvergenci znacime $f_n \to f$ (nebo $f_n(x) \to f(x)$ (nebo $\lim_{n\to\infty} f_n(x) = f(x)$))
2) Jesltize $\exists \lim f_n = f$, pak tato limitni funkce je urcena jednoznacne.
3) Zajima nas, ktere vlastnosti funkci jsou zachovany bodovou konvergenci (SKORO NIC)

EXAMPLE 9.1.3. (i) $f_n(x) = x^n, x \in [0,1]$
Fixujeme x, $\lim_{n\to\infty} f_n(x) = ?$
jestlize $x \in [0,1) \Rightarrow \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} x^n = 0$
jestlize $x = 1 \Rightarrow \lim_{n\to\infty} f_n(x) = 1$
$\Rightarrow f_n \to f,\ f = \begin{matrix} 0 \text{ na } [0,1) \\ 1 \text{ v bode } 1 \end{matrix}$
$\Rightarrow$ nezachovava se spojitost
zachovava se vzdy omezenost?
(ii) $f_n(x) = \begin{matrix} n \text{ na } \left(0,\frac{1}{n}\right) \\ \frac{1}{x} \text{ na } \left[\frac{1}{n},\infty\right) \end{matrix}$
MA40
$\forall n$ je f_n omezena, ale f omezena neni.
(iii) def $f_n(x) = \begin{matrix} \frac{n}{2} \text{ na } \left(-\frac{1}{n},\frac{1}{n}\right) \\ 0 \text{ na } \left[-1,-\frac{1}{n}\right] \cup \left[\frac{1}{n},1\right] \end{matrix}$



EXAMPLE 9.1.4. $(R)\int f_n(x)\,dx = 1$

(iv) def $f_n(x) = \begin{cases} \frac{n}{2} \text{ na } \left(-\frac{1}{n}, \frac{1}{n}\right) \setminus \{0\} \\ 0 \text{ na } \left[-1, -\frac{1}{n}\right] \cup \left[\frac{1}{n}, 1\right] \cup \{0\} \end{cases}$

$f_n \to f$, kde $f(x) == 0$ na $[1, 1]$

$\int f_n = 1 \forall n$, ale $\int f = 0$

$\Rightarrow$bodova konvergence je nanic.

DEFINITION 9.1.5. **(klicovy pojem)** Rekneme, ze posloupnost funkci $f_n : M \to \mathbf{R}$ **konverguje stejnomerne** k funkci f na M, jeslize

$\forall \varepsilon \exists n_0 \forall n \geq n_0 \forall x \in M : |f_n(x) - f(x)| < \varepsilon$

Rekneme, ze posloupnost funkci $f_n : M \to \mathbf{R}$ **konverguje lokalne stejnomerne** k funkci f na M, jeslize

$\forall x \exists r > 0$ tak, ze f_n konverguji stejnomerne na mnozine $M \cap B(x, r)$

Znaceni: $f_n \underset{\to}{\to} f, f_n \overset{loc.}{\underset{\to}{\to}} f$

THEOREM. *1. (stejnomerna limita spojitych funkci) TV*

*/*easy, deprecated, z pouzitim V2 je lehci*/*

Necht f_n konverguji stejnomerne k funkci f na $M \subset (P, \rho)$ a necht f_n jsou spojite $\forall n \in N$

Pak f je spojita na M.

PROOF. Vime: $\forall \varepsilon \exists n_0 \forall x \in M : |f_{n_0}(x) - f(x)| < \varepsilon$

Dale vime, ze f_{n_0} je spojita. Tedy $\forall x \in M \forall \varepsilon \exists \delta : \forall y : |x - y| < d \Rightarrow |f(x) - f(y)| < \varepsilon$

Zvolim $\varepsilon \Rightarrow \exists n_0, \delta$
pro $|x-y|<\delta$ mame:
$|f(x)-f(y)| \geq |f(x)-f_{n_0}(x)| + |f_{n_0}(x)-f_{n_0}(y)| + |f_{n_0}(y)-f(y)| < \varepsilon+\varepsilon+\varepsilon$ □

EXAMPLE 9.1.6. -"prehazovani limit"
$f_n = x^n, x \in [0,1]$
$\lim_{n\to\infty} \lim_{x\to 1-} x^n = \lim_{n\to\infty} 1 = 1$
$\lim_{x\to 1-} \lim_{n\to\infty} x^n == \lim_{x\to 1-} 0 = 0$
-obecne nelze prehazovat limity

THEOREM. *(ekvivalentni charakterizace stejnomerne konvergence) LV*

$f_n \underset{\rightarrow}{\overset{\rightarrow}{}} f$ *na* $M \Leftrightarrow \lim_{n\to\infty} \sup_{x\in M} |f_n(x)-f(x)| = 0$

PROOF. $f_n \underset{\rightarrow}{\overset{\rightarrow}{}} f \Leftrightarrow \forall\varepsilon\exists n_0 \forall n \geq n_0 \forall x \in M : |f_n(x)-f(x)| \leq \varepsilon$ (ekvivalentni)
$\Leftrightarrow \forall\varepsilon\exists n_0 \forall n \geq n_0 : \sup_{x\in M} |f_n(x)-f(x)| \leq \varepsilon$
(def. limity posloupnosti)$\Leftrightarrow \lim_{n\to\infty} \sup_{x\in M} |f_n(x)-f(x)| = 0$ □

THEOREM. *2. (Moore-Osgood) TV*
Necht (P,ρ) *je metricky prostor. Necht* $x_0 \in P$, *necht* $f_n, f : (P,\rho) \to \mathbf{R}$
Necht
(i) $f_n \underset{\rightarrow}{\overset{\rightarrow}{}} f$ *na* P
(ii) $\forall n \in \mathbf{N} : \lim_{x\to x_0} f_n(x) = a_n \in \mathbf{R}$
Potom
$\exists$ *limity* $\lim_{n\to\infty} a_n$ *a* $\lim_{x\to x_0} f(x)$ *a rovnaji se.*
(to jest $\lim_{n\to\infty} \lim_{x\to x_0} f_n(x) = \lim_{x\to x_0} \lim_{n\to\infty} f_n(x)$*)*

PROOF. /*hm. once more - vyznat se v tom + BC, zbytek - druha limita by slo (nerovnosti)
Zvolime ε
Dle (i) a Lemmatu $\exists n_0 : \sup_{x\in M} |f_{n_0}(x)-f(x)| < \varepsilon$
Dale k tomuto ε a n_0 existuje (dle (ii) pro $n=n_0$) nejake $r>0$ takove, ze
$x \in B(x_0,r) \Rightarrow |f_{n_0}(x)-a_{n_0}| < \varepsilon$
Vime: pro $n,m \geq n_0$ je $\sup_{x\in M} |f_n(x)-f_m(x)| < \varepsilon$
Tedy take $|a_m-a_n| \leq \varepsilon$
Tudiz z B-C podminky: $\exists \lim_{n\to\infty} a_n = a \in \mathbf{R}$
Musime dokazat, ze $a = \lim_{x\to x_0} f(x)$
Pro $x \in B(x_0,r)$ mame:
$|f(x)-a| \leq |f(x)-f_{n_0}(x)| + |f_{n_0}(x)-a_{n_0}| + |a_{n_0}-a| \leq \varepsilon+\varepsilon+\varepsilon$
Tedy $\forall\varepsilon\exists r : |f(x)-a| \leq 3\varepsilon \Rightarrow \lim_{x\to x_0} f(x) = a$ □

NOTE 9.1.7. Z V2 plyne jednoduchy dukaz V1:
$\lim_{x\to x_0} f(x) = \lim_{x\to x_0} \lim_{n\to\infty} f_n(x) \overset{\rightarrow\rightarrow}{=} \lim_{n\to\infty} f_n(x) \lim_{x\to x_0} f_n(x) \overset{\text{spojitost } f_n}{=} \lim_{n\to\infty} f_n(x_0) \overset{(f_n\to f)}{=} f(x$

-

NOTE 9.1.8. (netrivialni)
$f_n \overset{loc}{\underset{\rightarrow}{\rightarrow}} f$ na $(a,b) \Leftrightarrow \forall [c,d] \subset (a,b) : f_n \underset{\rightarrow}{\overset{\rightarrow}{}} f$ na $[c,d]$

EXAMPLE 9.1.9. $f_n(x) = x^n$ neni $f_n \overset{loc}{\underset{\rightarrow}{\rightarrow}}$ na $[0,1]$

ale je $f_n \overset{loc}{\underset{\rightarrow}{\rightarrow}}$ na $[0,1)$, tedy $x^n \underset{\rightarrow}{\overset{\rightarrow}{}} 0$ na kazdem $[0,a], a<1$

NOTE 9.1.10. Posloupnost f_n je na M stejnomerne konvergentni prave kdyz je na mnozine M **stejnomerne Cauchyovska**, to jest
$\forall\varepsilon\exists n_0 \in \mathbf{N} : \forall m,n \geq n_0 \forall x \in M \, |f_n(x)-f_m(x)| < \varepsilon$

THEOREM. *3. (zamena limity a derivace) TV*
Necht f_n *maji na* (a,b) *vlastni derivace* $f_n\prime$.
Necht
(i) $f_n\prime \overset{loc}{\underset{\rightarrow}{\rightarrow}} g$ *na* (a,b)
(ii) $\exists x_0 \in (a,b) : \{f_n(x_0)\}$ *je konvergentni posloupnost*

Potom $\exists f$ takova, ze $f_n \overset{loc}{\underset{\rightarrow}{\rightarrow}} f$, a navic $f\prime = g$ na (a,b)

PROOF. /*$f_n \overset{loc}{\underset{\rightarrow}{\rightarrow}}$ - 3uhelnik + velmi pekne pouziti Lagrange na rozdil
$f\prime = g$ - prohozeni limit*/
Zvolime $[c,d] \subset (a,b)$ tak, aby $x_0 \in [c,d]$
Chceme: $f_n \overset{\rightarrow}{\rightarrow}$ na $[c,d]$
Zvolime ε. Potom $\exists n_0 \forall m, n \geq n_0$
$|f_n(x_0) - f_m(x_0)| < \varepsilon$ (z predp. (ii))
&
$|f_n\prime(x) - f_m\prime(x)| < \varepsilon \forall x \in [c,d]$
Mame pro $x \in [c,d]$
$|f_n(x) - f_m(x)| \leq |(f_n(x) - f_m(x)) - (f_n(x_0) - f_m(x_0))| + |f_n(x_0) - f_m(x_0)|$
nebo (x, x_0)
f_n, f_m jsou spojite
$\exists \xi \in (x_0, x)$
$|(f_n(x) - f_m(x)) - (f_n(x_0) - f_m(x_0))| \overset{\text{(podle V Lagrange pro } f_n - f_m)}{=} |(f_n\prime(\xi) - f_m\prime(\xi))(x - x_0)|$
$\Rightarrow |f_n(x) - f_m(x)| \leq (f_n\prime(\xi) - f_m\prime(\xi))(d-c) + \varepsilon \leq \varepsilon(d-c) + \varepsilon$
Tedy $f_n \overset{\rightarrow}{\rightarrow}$ na intervalu $[c,d]$

To znamena, ze $\exists f$ na $[c,d]$ takova, ze $f_n \overset{\rightarrow}{\rightarrow} f$
Musime dokazat, ze $f\prime = g$
Necht $z \in [c,d]$. Potom
$f\prime(x) = \lim_{h\to 0} \frac{f(z+h)-f(z)}{h} = \lim_{h\to 0} \lim_{n\to\infty} \frac{f_n(z+h)-f_n(z)}{h}$
$\overset{\text{stejnomerna na int.}}{=} \lim_{n\to\infty} \lim_{h\to 0} \frac{f_n(z+h)-f_n(z)}{h} = \lim_{n\to\infty} f_n\prime(z) = g(z)$
Tedy $f\prime = g$ □

EXAMPLE 9.1.11. k V3:
bez predpokladu (ii) veta neplati: f_n konstantni, $f_n == c_n$, $\{c_n\}$ diverguje.
Pak $f_n\prime == 0, f_n\prime \overset{\rightarrow}{\rightarrow} 0$, ale f_n nekonverguje

THEOREM. *4. (stejnomerna limita newtonovsky integrovatelnych funkci) TV*
Necht $f_n \overset{\rightarrow}{\rightarrow} f$ na (a,b)
Necht $f_n \in \mathcal{N}(a,b)\, \forall n$
Potom $f \in \mathcal{N}(a,b)$ a $(N)\int_a^b f(x)\,dx = \lim_{n\to\infty}(N)\int_a^b f_n(x)\,dx$
(t.j. $\int \lim f_n = \lim \int f_n$)

PROOF. /*fixnout jednim bodem (kontanta $\int$) + V3 + $F(b-)$ - prehozeni limit*/
$f_n \in \mathcal{N}(a,b) \Rightarrow \exists F_n$ prim. fce ($F_n\prime = f_n$ na (a,b))
Zvolime $x_0 \in (a,b)$. Pak lze volit F_n tak, aby $F_n(x_0) = 0 \forall n \in \mathbf{N}$
Potom F_n maji vlastni derivace na (a,b)
$F_n\prime = f_n \overset{\rightarrow}{\rightarrow} f$ na (a,b)
$F_n(x_0) = 0 \Rightarrow F_n(x_0) \to 0$
$\overset{V3}{\Rightarrow} \exists F$ takova, ze $F_n \overset{\rightarrow}{\rightarrow} F$ na (a,b) a $F\prime = f$
Potom $F(b-) = \lim_{x\to b-} F(x) = \lim_{x\to b-} \lim_{n\to\infty} F_n(x) = \lim_{n\to\infty} \lim_{x\to b-} F_n(x) = \lim_{n\to\infty} F_n(b-)$
& to same v $a+$
a tedy $(N)\int_a^b f(x)\,dx = F(b-) - F(a+) = \lim_{n\to\infty}(F_n(b-) - F_b(a+)) = \lim_{n\to\infty}(N)\int_a^b f_n(x)\,dx$ □

9.1.1. informativni vety bez dukazu.

THEOREM. *5. (Dini)*
Necht $f_n \to f$ na kompaktni mnozine, necht konvergence je monotonni, necht f_n i f jsou spojite.
Pak $f_n \overset{\rightarrow}{\rightarrow} f$

(BEZ DUKAZU)

THEOREM. *6. (Weierstrass)*

Necht $f \in \mathcal{C}([a,b])$. *Potom* $\exists$*posloupnost polynomu* P_n *na* $[a,b]$ *takova, ze* $P_n \underset{\to}{\to} f$ *na* $[a,b]$

(BEZ DUKAZU)

9.2. rady funkci

DEFINITION 9.2.1. **(klicovy pojem)** Rekneme, ze rada funkci $\sum_{k=1}^{\infty} f_k(x)$ konverguje **bodove** ((lokalne) **stejnomerne**) jestlize posloupnost $\{\sum_{k=1}^{n} f_k(x)\}_{n=1}^{\infty}$ konverguje bodove ((lokalne) stejnomerne)

THEOREM. *7. (Weierstrassovo kriterium konvergence rad) LV*
*/*nejjednosussi priklad v pisemnce - jedinej zpusob jak resit*/*
Necht f_n *jsou def. na* (a,b) *a necht plati:*
$\forall n : \sup_{x\in(a,b)} |f_n(x)| \le a_n \& \sum a_n < \infty$
Potom $\sum_{n=1}^{\infty} f_n(x)$ *konverguje stejnomerne na* (a,b)

PROOF. Zvolime ε. Pak $\exists n_0 : \forall m,n \ge n_0 : \left|\sum_{k=n+1}^{m} a_k\right| < \varepsilon$ (predp˙ $\sum a_n < \infty$)
Zvolime $m, n \ge n_0$, pak (napr. $n > m$)
$\left|\sum_{k=1}^{n} f_k(x) - \sum_{k=1}^{m} f_k(x)\right| = \left|\sum_{k=m+1}^{n} f_k(x)\right| \le \sum_{k=m+1}^{n} |f_k(x)| < \varepsilon$ /*toto bylo blbe zapsany - check it out*/ □

THEOREM. *8 (derivovani rad)*

Necht f_n *maji vlastni derivace na* (a,b), *necht* $\sum f_n\prime \underset{\to}{\overset{loc}{\to}}$ *na* (a,b). *Necht* $\exists x_0 \in (a,b) : \sum f_n(x_0)$ *konverguje.*

Potom $\sum_{n=1}^{\infty} f_n \underset{\to}{\overset{loc}{\to}}$ *na* M *a navic* $\left(\sum_{n=1}^{\infty} f_n(x)\right)\prime = \sum_{n=1}^{\infty} (f_n\prime(x))$

PROOF. plyne ihned z V3 □

EXAMPLE 9.2.2. $\sum_{n=1}^{\infty} (-1)^n \frac{x^{n+1}}{2n+1}$ vysetrit konvergenci na $\mathbf{R}$
Kde konverġuje bodove?
zrejme konverguje $|x| < 1$
diverguje $|x| > 1$
$\sup_x |f_n(x)| = \frac{1}{2n+1}$ (Ws nejde pouzit)
$\delta \in (0,1)$
$\sup_{x\in[-\delta,\delta]} |f_n(x)| = \frac{\delta^{2n+1}}{2n+1} = s_n, \sum s_n < \infty \overset{Ws}{\Rightarrow} \sum f_n \underset{\to}{\to}$ na $[-\delta,\delta]$
$\Rightarrow \sum f_n \underset{\to}{\overset{loc}{\to}}$ na $[-1,1]$
jinak: $f_n(x) = (-1)^n \frac{x^{n+1}}{2n+1}, f_n\prime(x) = (-1)^n x^{2n}$
chceme: $\sum (-1)^n x^{2n} \underset{\to}{\overset{loc}{\to}}$ na $(-1,1)$
$\delta \in (0,1)$, pak
$\sup_{x\in[-\delta,\delta]} (-1)^n x^{2n} = \delta^{2n}$
$\sum s_n < \infty \overset{Ws}{\Rightarrow} \sum f_n\prime \underset{\to}{\to}$ na $[-\delta,\delta]\,\forall\delta$
$\overset{Ws}{\Rightarrow} \sum f_n\prime \underset{\to}{\to}$ *loc* na $(-1,1)$
chceme najit $x_0 : \sum f_n(x_0)$ konverguje
poloz $x_0 = 0$, pak zrejme $\sum f_n(x_0) = 0$
$\overset{V8}{\Rightarrow} \sum f_n \underset{\to}{\to}$ *loc* na $(-1,1)$ (to uz vime) & $(\sum f_n)\prime = \sum f_n\prime$
Vime: $\sum_{n=0}^{\infty} f_n\prime(x) = \sum_{n=0}^{\infty} (-1)^n x^{2n} = \frac{1}{1+x^2}$ pro $\forall x \in (-1,1)$
tedy polozime-li $F(x) = \sum f_n$, pak $F\prime(x) = \frac{1}{1+x^2} \Rightarrow f(x) = \arctan(x) + c$
Vime, ze $f(0) = 0 \Rightarrow c = 0$
$\sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{2n+1} = \arctan x$ pro $x \in (-1,1)$
$\Rightarrow \arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$
$x = 1$ - $\frac{\pi}{4} = \dots$

NOTE 9.2.3. (o konvergencich)
pr. $\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x \forall x \in \mathbf{R}$ (koverguje bodove)
nekonverguje stejnomerne (chyba je radu $\frac{x}{nn!}$)
Ale konverguje loc. stejnomerne a absolutne. $\sum_{n=0}^{\infty} \left|\frac{x^n}{n!}\right| = e^{|x|}$
Absolutni konvergence obecne neimplikuje stejnomernou.
Ani obracene. (cviceni)
Weierstrassovo kriterium lze pouzit **jen** v pripadech absolutni konvergence.

Theorem. *9. (Abelovo kriterium)*
Necht $\sum_{n=1}^{\infty} f_n$ stejnomerne konverguje na mnozine M
Necht $\{g_n\}$ je ***stejnomerne omezena*** *na M ($\exists K \forall n \forall x \in M : |g_n(x)| \leq K$. Necht $\forall x \in M$ je $\{g_n(x)\}$ monotonni (bud rostouci nebo klesajici) /*muze v kazdym bode jinak*/*

Potom $\sum f_n g_n \overset{\rightarrow}{\rightarrow}$ na M

Proof. Pro $N \in \mathbf{N}$ oznacme $s_N = \sum_{n=1}^{N} f_n(x), t_N = \sum_{n=N+1}^{\infty} f_n(x)$
Pak (vynechavam argument x)
jedinny trik: $f_n g_n = -t_n [g_n - g_{n+1}] + t_{n-1} g_n - t_n g_{n+1}$
$/ \sum_{n=M}^{N}$
$\sum_{n=M}^{N} f_n g_n = \sum_{n=M}^{N} (-t_n [g_n - g_{n+1}]) + t_{M-1} g_M - t_N g_{N+1}$
Zvolime ε. Pak $\exists M : \forall x \forall n \geq M-1 : |t_n(x)| < \varepsilon$ (protoze $\sum f_n \overset{\rightarrow}{\rightarrow}$)
a navic $|g_n(x)| \leq K \forall n, x$
Dale: $\sum_{i=M}^{N} |g_n(x) - g_{n+1}(x)| = |g_M(x) - g_{N+1}(x)|$(z monotonie)
Celkem: $\left|\sum_{n=M}^{N} f_n(x) g_n(x)\right| \leq \sum_{n=M}^{N} |t_n(x)| |g_n(x) - g_{n+1}(x)| + |t_{M-1}(x)| |g_M(x)| + |t_N(x)| |g_{N+1}(x)|$
$< \varepsilon \sum_{n=M}^{N} |g_n(x) - g_{n+1}(x)| + \varepsilon K + \varepsilon K$
$= \varepsilon (g_M(x) - g_{N+1}(x)) + \varepsilon K + \varepsilon K \leq \varepsilon (|g_M(x)| + |g_{N+1}(x)|) + 2\varepsilon K \leq 4K\varepsilon$
Tedy $\sum f_n(x) g_n(x)$ je stejnomerne Cauchyovska a tedy stejnomerne konvergentni. □

Theorem. *10. (Dirichletovo kriterium)*
Necht $\{\sum_{n=1}^{\infty} f_n\}$ je stejnomerne omezena (tj. $\sum f_n$ ma stejnomerne omezene castecne soucty) na M
*Necht $g_n \overset{\rightarrow}{\rightarrow} 0$ a je $\{g_n(x)\}$ monotonni (bud rostouci nebo klesajici) /*muze v kazdym bode jinak*/*

Potom $\sum f_n g_n \overset{\rightarrow}{\rightarrow}$ na M

Proof. Pro $N \in \mathbf{N}$ oznacme $s_N = \sum_{n=1}^{N} f_n(x)$
Pak $f_n(x) g_n(x) = s_n(x) [g_n(x) - g_{n+1}(x)] - s_{n-1}(x) g_n(x) + s_n(x) g_{n+1}(x)$
$/ \sum_{n=M}^{N}$:
$\sum_{n=M}^{N} f_n(x) g_n(x) = \sum_{n=M}^{N} s_n(x) [g_n(x) - g_{n+1}(x)] - s_{M-1}(x) g_M(x) + s_N(x) g_{N+1}(x)$
$\Rightarrow \left|\sum_{n=M}^{N} f_n(x) g_n(x)\right| \leq \sum_{n=M}^{N} |s_n(x)| |g_n(x) - g_{n+1}(x)| + |s_{M-1}(x)| |g_M(x)| + |s_N(x)| |g_{N+1}(x)|$
Vime: $|s_n(x)| \leq K$ (stejnomerna omezenost castecnych souctu)
Dale: $\sum_{n=M}^{N} [g_n(x) - g_{n+1}(x)] = |g_M(x) - g_{N+1}(x)|$ (monotonie)
ke zvolenemu $\varepsilon > 0 \exists M \forall n \geq M \forall x \, |g_n(x) < \varepsilon|$
Tedy $\left|\sum_{n=M}^{N} f_n(x) g_n(x)\right| \leq K \sum_{n=M}^{N} |g_n(x) - g_{n+1}(x)| + K\varepsilon + K\varepsilon$
$= K |g_M(x) - g_{N+1}(x)| + 2K\varepsilon \leq 4K\varepsilon$ □

Example 9.2.4. $\sum_{n=1}^{\infty} \frac{\sin nx}{n}, x \in [0, 2\pi]$
Elem˙geometrie + indukce $= \sum_{n=1}^{N} \sin(nx) = \frac{\sin\left(\frac{nx}{2}\right) + \sin\left(\frac{(n+1)x}{2}\right)}{\sin\frac{x}{2}}$
Pro $\delta \in (0, \pi)$, pak na $[\delta, 2\pi - \delta]$: $\left|\sum_{n=1}^{N} \sin(nx)\right| \leq \frac{1}{\sin\frac{x}{2}}$
Def $f_n(x) = \sin(nx)$, pak $\sum f_n$ ma om. cast. soucty (stejnomerne na $[0, 2\pi]$)
Def $g_n(x) = \frac{1}{n}$ ($g_n \overset{\rightarrow}{\rightarrow} 0, g_n \downarrow\downarrow 0$)

$\overset{Dirichlet}{\Rightarrow} \sum_{n=1}^{\infty} \frac{\sin(nx)}{n} \overset{\rightarrow}{\rightarrow}$ *loc* na $[0, 2\pi]$

Note 9.2.5. $\sum_{n=1}^{\infty} \frac{\sin(nx)}{n} = \begin{cases} 0 & x \in \{0, 2\pi\} \\ \frac{\pi - x}{2} & x \in (0, 2\pi) \end{cases}$

Example 9.2.6. $\sum_{n=1}^{\infty} (-1)^n g_n(x) \overset{\rightarrow}{\rightarrow}$ pokud $g_n \overset{\rightarrow}{\rightarrow} 0, g_n(x)$ monotonni
$\sum_{n=1}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$
$g_n(x) = \frac{x^{2n+1}}{2n+1}$
na $[-\delta, \delta], \delta \in (0, 1) : \sup_{x \in [-\delta, \delta]} |g_n(x)| = \frac{\delta^{2n+1}}{2n+1}$
$\Rightarrow g_n \overset{\rightarrow}{\rightarrow} loc 0, g_n \downarrow\downarrow 0 \Rightarrow \sum \overset{\rightarrow}{\rightarrow}$ *loc.*

APPENDIX A

Shrnuté pomůcky k písemce

Pokud to máte dovolené, vezměte si je na písemku.

A.1. pomucky

A.1.1. vzorce. $\cos^2\alpha = \frac{1+\cos(2\alpha)}{2}$
$\left|\sin\frac{\alpha}{2}\right| = \sqrt{\frac{1-\cos\alpha}{2}}$
$\left|\cos\frac{\alpha}{2}\right| = \sqrt{\frac{1+\cos\alpha}{2}}$
$\cos x = \sqrt{\frac{1}{\tan^2 x}}$

A.1.2. integrovani.

tabulkove integraly.

$f(x)$	$\int -c$	$x \in$		
$n^n, n \in \mathbf{Z}\setminus\{-1\}$	$\frac{x^{n+1}}{n+1}$	$x \in \mathbf{R}$ pokud $n \geq 0$,$x \in \mathbf{R}\setminus\{0\}$ pokud $n < 0$		
x^{-1}	$\log\|x\|$	$x \neq 0$		
e^x	e^x	$\mathbf{R}$		
$\sin x$	$-\cos x$	$\mathbf{R}$		
$\cos x$	$\sin x$	$\mathbf{R}$		
$\frac{1}{\cos^2 x}$	$\tan x$	$\left(-\frac{\pi}{2};\frac{\pi}{2}\right)+k\pi, k \in \mathbf{Z}$		
$\frac{1}{\sin^2 x}$	$-cotg(x)$	$(0;\pi)+k\pi, k \in \mathbf{Z}$		
$\frac{1}{1+x^2}$	$\arctan x$	$\mathbf{R}$		
$\frac{1}{\sqrt{1-x^2}}$	$\arcsin x = -\arccos x$	$(-1,1)$		

substituce.

na co	substituce	diferencial	vyjadreni	
$R(\sin x, \cos x)$	$t = \tan\frac{x}{2}$	$dx = \frac{2dt}{1+t^2}$	$\cos x = \frac{1-t^2}{1+t^2}$	$\sin x = \frac{2t}{1+t^2}$
$R(\sin x, \cos x) = R(-\sin x, -\cos x)$	$y = \tan x$	$dx = \frac{1}{1+y^2}dy$	$\cos x = \frac{1}{\sqrt{1+y^2}}$	$\sin x = \frac{y}{\sqrt{1+y^2}}$
$R\left(x, \left(\frac{Ax+B}{Cx+D}\right)^{\frac{1}{q}}\right), q \in N, q > 1$	$y = \left(\frac{Ax+B}{Cx+D}\right)^{\frac{1}{q}}$	$dx = qy^{q-1}\frac{DA-CB}{(A-Cy^q)^2}dy$	$x = \frac{Dy^q-B}{(A-Cy^q)}$	
$R\left(x, \sqrt{ax^2+bx+c}\right), a > 0$, ired.	$\sqrt{ax^2+bx+c} = \sqrt{a}x + y$ (Eu)			

rozklad na parcialni zlomky. $Q(x) = a_n(x-x_1)^{\alpha_1}\dots(x-x_k)^{\alpha_k}\left(x^2+p_1x+q_1\right)^{\beta_1}\dots\left(x^2+p_lx+q_l\right)^{\beta_l}\ \forall x \in \mathbf{R}$
tak, ze $\frac{P(x)}{Q(x)} = \frac{A_1^1}{x-x_1}+\dots+\frac{A_{\alpha_1}^1}{(x-x_1)^{\alpha_1}}+\dots+\frac{A_1^k}{x-x_k}+\dots+\frac{A_{\alpha_k}^k}{(x-x_k)^{\alpha_k}}$
$+\frac{B_1^1x+C_1^1}{x^2+p_1x+q_1}+\dots+\frac{B_{\beta_1}^1+C_{\beta_1}^1}{(x^2+p_1x+q_1)^{\beta_1}}+\dots+\frac{B_1^lx+C_1^l}{x^2+p_lx+q_l}+\dots+\frac{B_{\beta_l}^l+C_{\beta_l}^l}{(x^2+p_lx+q_l)^{\beta_l}}$

A.1.3. mnohem VICE promennych.
chain rule. $\mathbf{R^s} \overset{g}{\to} \mathbf{R^n} \overset{f}{\to} \mathbf{R}$, $\exists$ diferencialy
pak:
$\frac{\delta(f\bullet g)}{\delta x_i}(a) = \sum_{j=1}^n \frac{\delta f}{\delta y_j}(g(a))\frac{\delta g_j}{\delta x_i}(a)$ pro $i = 1,\dots,s$

A.2. kucharky

A.2.1. urcovani definitnosti matice. (A_k - hlavni subdeterminant matice A radu k)
PD $\Leftrightarrow$vsechny hlavni subdeterminanty > 0 ($|A_k|$
ND $\Leftrightarrow |A_k|(-1)^k > 0 \forall k = 1,\dots,n$
IN - $k = 2 : A_2 < 0$
$k = 3$?? ?????????

A.2.2. implicitni funkce. $(F_i(x_1,\dots,x_n,y_1,\dots y_m))$
overeni predpokladu:

(1) $F_1,\dots,F_m \in \mathcal{C}^1$ na okoli
(2) $\frac{\delta(F_1,\dots,F_m)}{\delta(y_1,\dots,y_m)} \neq 0$

vypocet: $\frac{\delta y_i}{\delta x_j} = -\frac{\frac{\delta(F_1,\dots,F_m)}{\delta(y_1,y_{i-1},x_j,y_{i+1},\dots,y_m)}}{\frac{\delta(F_1,\dots,F_m)}{\delta(y_1,\dots,y_m)}}$ (Kramerovo pravidlo)

pozn. $\left|\frac{\delta(F_1,\dots,F_m)}{\delta(y_1,\dots,y_m)}\right| = \begin{vmatrix} \frac{\delta F_1}{\delta y_1} & \cdots & \frac{\delta F_1}{\delta y_m} \\ \vdots & \ddots & \vdots \\ \frac{\delta F_m}{\delta y_1} & \cdots & \frac{\delta F_m}{\delta y_m} \end{vmatrix}$

A.2.3. vazane extremy.

kandidati:

(1) stacionarni body ($Df(a) = o, a \in$ vnitrku M $(M^\bullet)$
(2) singularni body ($Df(a)$ neexistuje)
(3) hranicni body

1. najit; test pomoci $D^2 f(a)$
2. porovnat hodnoty funkce v singularnich bodech s ostatnimi
3. bud to jde hlavou (parametrizovat hranici), pak hledam volnej extrem
   nebo Lagrangeovy multiplikatory

4 - postacujici podminka: !!! dulezite to vysvetlit
Veta o nabyvani extremu na kompaktu
(kompakt v $\mathbf{R^n}$ $\Leftrightarrow$omezena & uzavrena)
pokud to neni kompakt:
rozdelim na kompakt a zbytek a ten zbytek nak odhadnu
napr: najdu r takove, ze na $\left\{[x,y,z] : x^2+y^2+z^2 > r^2\right\}$ je $f(x,y,z) \geq R$ - odhad treba $|x| \leq \sqrt{x^2+y^2+z^2}$

A.2.4. stejnomerna konvergence. $f_n \overset{\rightarrow}{\rightarrow} f \Leftrightarrow \sup_{x\in M} |f_n(x) - f(x)| \to 0$

(1) najdu bodovou limitu $f(x) = \lim_{n\to\infty} f_n(x) \, \forall x \in M$
(2) polozim $g_n(x) = f_n(x) - f(x)$, najdu $\sup_{x\in M} |g_n(x)| = s_n$
(3) $f_n \overset{\rightarrow}{\rightarrow} \Leftrightarrow s_n \to 0$

A.2.5. stejnomerna konvergence rad. spocitam $\sup_{x\in[a,b]} |f_n(x)| = s_n$
(Weierstrass) kdyz $\forall n : s_n \leq a_n \& \sum a_n < \infty$, potom i puvodni rada konverguje stejnormerne na danem intervalu
lokalni stejnomernost:
kdyz $\forall \delta \in \max\{a,b\} : \sum f_n \overset{\rightarrow}{\rightarrow}$ pro $\forall x \in (-\delta, \delta)$, potom rada konverguje lokalne stejnomerne na $[a,b]$
$\sum f_n g_n$ na G
(Abel)
$\sum f_n \overset{\rightarrow}{\rightarrow}$ & g_n je stejnomerne omezena $(\exists K \forall n \forall x : |g_n| < K)$ & $\forall x$ je $g_n(x)$ monotonni (pro kazde jinak)

potom $\sum f_n g_n \overset{\rightarrow}{\rightarrow}$ na G
(Di)
$\sum f_n$ ma stejnomerne omezene castecne soucty (tedy $\{\sum_{n=1}^{\infty} f_n\}$ je stejnomerne omezena)
$\& g_n \overset{\rightarrow}{\rightarrow} 0$ & $\forall x$ je $g_n(x)$ monotonni (pro kazde jinak)

potom $\sum f_n g_n \overset{\rightarrow}{\rightarrow}$ na G

APPENDIX B

Poslední rady pred zkouskou :-)

(ponecháno vlastně v původním znění)

B.0.6. integral. /*nechat na konec*/
primitivni funkce (konstanty, interval (kde integral $\exists$), lepeni)
urcity integral (Newtonuv)
aplikace: delka krivky, objem a povrch telesa
techniky: per, 2subst
zakladni substituce na racionalni funkci
lepeni
!integrace racionalnich funkci

B.0.7. konvergence posloupnosti a rad funkci.
lehci: Weistrass ($\Rightarrow \sum f_n \underset{\rightarrow}{\rightarrow}$)

Dirichlet ((Abel)) ($\Rightarrow \sum (-1)^n g_n \underset{\rightarrow}{\rightarrow}$)

zakuklene: $F(x) = \sum f_n(x)$, zjistit, zda je spojita
$F\prime(0)$
pozor! u obou pripadu staci lokalni stejnomernost
tezsi: (spis tam nebudou)
Moore-Ozgood
$\lim_{x\to x_0} \lim_{n\to\infty} f_n(x)$
nebo $\lim_{x\to x_0} \sum f_n$
u obou $f_n \underset{\rightarrow}{\rightarrow} f \Leftrightarrow [\sup_x |f_n - f| \to 0]$

B.0.8. extremy.
volne. (pozor na globalni)
techniky: stacionarni body, test $D^2 f$
vazane. techniky: param hranici nebo Lagrangeovy multiplikatory
+ postacujici podminka pro existenci extremu (skoro vzdy $\exists$ extremu spojite funkce na intervalu)
uskali. slovni zadani: vzdalenost M od pocatku, nejmensi koule (maximum)
kombinace volnych a vazanych extremu: $M = \{g(x, \ldots) \le 1\}$ - stacionarni body + Langange
2 vazby: $\frac{\delta f}{\delta x} + \lambda \frac{\delta F}{\delta x} + \phi \frac{\delta G}{\delta x}$

B.0.9. implicitni funkce. - zadne uskali nejsou
rozdil - vice vazeb ($\Rightarrow$vice implicitnich funkci)
jedna nezavisla promenna $\Rightarrow$ derivace jsou neparcialni

APPENDIX C

Vzorově řešená písemka

C.1. řešená písemka

F1. $\int x \log\left(1-\frac{x^2}{2}\right) dx$
$y = 1-\frac{x^2}{2}$
$dy = -x dx$
$(1-x^2>0 \Leftrightarrow x \in \left(-\sqrt{2},\sqrt{2}\right))$
$= \int \log(y)\, dy = -y\left(\log(y)+1\right)+c = -\left(1-\frac{x^2}{2}\right)\left(\log\left(1-\frac{x^2}{2}\right)+1\right)+c$

F2. $\sum_{n=2}^{\infty}(-1)^n \frac{1+nx}{n^2-x} \overset{?}{\rightrightarrows}$ na $(0,2)$?

Dirichletovo:
(i) $\left\{\left|\sum_{n=1}^{N} f_n\right|\right\}$ stejnomerne omezena
(ii) $g_n \rightrightarrows 0$
(iii) $\{g_n(x)\}$ mon. pro $\forall x \in (0,2)$
$f_n = (-1)^n$
$g_n(x) = \frac{1+nx}{n^2-x^2}, x \in (0,2), n = 2, \ldots$
overeni:
$\left|\sum_{n=1}^{N}(-1)^n\right| = \begin{matrix}1\\0\end{matrix}$

$g_n(x) = \frac{1+nx}{n^2-x^2} \rightrightarrows 0 \Leftrightarrow \sup_{x\in(0,2)}|g_n(x)| \to 0$
hledam suprema $g_n(x)$
$g_n(x) \geq 0$ na $(0,2)$
$g_n\prime(x) = \frac{n\left(n^2-x^2\right)-(1+nx)(-2x)}{\left(n^2-x^2\right)^2} = \frac{n^3-nx^2+2x+2nx^2}{\left(n^2-x^2\right)^2} = \frac{n^3+2x+nx^2}{\left(n^2-x^2\right)^2} > 0$
$\Rightarrow g_n$ je rostouci na $(0,2) \Rightarrow \sup_{x\in(0,2)} g_n(x) = \lim_{x\to 2-} g_n(x)$
g_n spojita na $(0,2] \Rightarrow \lim_{x\to2-} g_n(x) = g_n(2) = \frac{1+2n}{n^2-4} \to 0 \Rightarrow g_n \rightrightarrows 0$
(iii) monotonie podle n - bud je videt, nebo se odectou/vydelej nasledujici
$g_n(x) - g_{n+1}(x) = \frac{1+nx}{n^2-x^2} - \frac{1+(n+1)x}{(n+1)^2-x^2} = \frac{(1+nx)\left((n+1)^2-x^2\right)-(1+(n+1)x)\left(n^2-x^2\right)}{\left(n^2-x^2\right)\left((n+1)^2-x^2\right)}$
citatel$=(1+nx)\left((n+1)^2-x^2\right) - (1+(n+1)x)\left(n^2-x^2\right)$
$= (n+1)^2 - x^2 + n(n+1)^2 x - nx^3 - n^2 + x^2 - n^2(n+1) + x^3(n+1)$
$= (n+1)^2 - n^2 + n(n+1)x(n+1-n) + x^3 = (n+1)^2 - n^2 + n(n+1)x + x^3 > 0 \forall x \in (0,2)\, \forall n \in \mathbf{N}$
$\overset{Dirichlet}{\Rightarrow}$ rada konverguje stejnomerne

F3. /*u volnych extremu v podstate neni prostor pro tvurci mysleni*/
najdete vsechny lokalni extremy f na $\mathbf{R}^2$ a rozhodnete o $\exists$ globalnich
$f(x,y) = x^3 - 3x + xy^2$
mohou nastat ve:

(1) stacionarnich bodech ($\nabla f = o$)
(2) singularnich (jedna z parcialnich derivaci neexistuje)
(3) na hranici dane oblasti (vazany extrem)

polynom 2 promennych$\Rightarrow$nema singularni body
$\mathbf{R}^2$ nema hranici
globalni extremy neexistuji, nebot pro $y=0$ je funkce neomezena ($\lim_{[x,y]\to[\pm\infty,0]} f(x,y) = \pm\infty$)
lokalni extremy (pokud vubec), jsou ve stacionarnich bodech
$\frac{\delta f}{\delta x} = 3x^2 - 3 + y^2 = 0$
$\frac{\delta f}{\delta y} = 2xy = 0$
$x = 0 \Rightarrow |y| = \sqrt{3}$
$y = 0 \Rightarrow |x| = 1$
4 body
postacujici podminka:
$\frac{\delta f}{\delta^2 x^2} = 6x$
$\frac{\delta f}{\delta x \delta y} = 2y$
$\frac{\delta f}{\delta^2 y^2} = 2x$
$D^2 f(x,y) = \begin{pmatrix} 6x & 2y \\ 2y & 2x \end{pmatrix}$
$det\left(D^2 f(x,y)\right) = 12x^2 - 4y^2$
$[0,\sqrt{3}] : \begin{pmatrix} 0 & 2\sqrt{3} \\ 2\sqrt{3} & 0 \end{pmatrix}$ indefinitni

$[0,-\sqrt{3}]:\begin{pmatrix}0 & -2\sqrt{3}\\ -2\sqrt{3} & 0\end{pmatrix}$ indefinitni

$[1,0]:\begin{pmatrix}6 & 0\\ 0 & 2\end{pmatrix}$ je PD $\Rightarrow$lokalni minimum -2

$[-1,0]:\begin{pmatrix}-6 & 0\\ 0 & -2\end{pmatrix}$ je ND $\Rightarrow$lokalni maximum 2

F4. ukazte, ze

$y=z^3-x-8$

$z^4=y^5-x^3+16$

definuji na okoli bodu 0 hladke funkce $y(x), z(x)$, pro ktere plati $y(0)=0, z(0)=2$ a vypoctete jejich derivaci v 0

$F(x,y,z)=+z^3-x-8-y=0$

$G(x,y,z)=y^5-x^3-z^4+16=0$

predpoklady: (i) $F,G\in\mathcal{C}^1$

(ii) $[x_0,y_0,z_0]\in M=\{[x,y,z],F(x,y,z)=G(x,y,z)=0\}$ - jenom dosadim:

$F(0,0,2)=$

(iii) $\frac{\delta(F,G)}{\delta(y,z)}(x_0,y_0,z_0)\neq 0$

$$\frac{\delta(F,G)}{\delta(y,z)}=\begin{vmatrix}\frac{\delta F}{\delta y} & \frac{\delta F}{\delta z}\\ \frac{\delta G}{\delta y} & \frac{\delta G}{\delta z}\end{vmatrix}=\begin{vmatrix}-1 & 3z^2\\ 5y^4 & -4z^3\end{vmatrix}\neq 0$$

pouziti. $\Rightarrow \exists y,z\in\mathcal{C}^1: y\prime(x_0)=-\frac{\frac{\delta(F,G)}{\delta(x,z)}}{\frac{\delta(F,G)}{\delta(y,z)}}$

$$y\prime(x_0)=-\frac{\begin{vmatrix}\frac{\delta F}{\delta x} & \frac{\delta F}{\delta z}\\ \frac{\delta G}{\delta x} & \frac{\delta G}{\delta z}\end{vmatrix}}{\begin{vmatrix}-1 & 3z^2\\ 5y^4 & -4z^3\end{vmatrix}}=-\frac{\begin{vmatrix}-1 & 3z^2\\ -3x^2 & -4z^3\end{vmatrix}}{\begin{vmatrix}-1 & 3z^2\\ 5y^4 & -4z^3\end{vmatrix}}\overset{[0,0,2]}{=}=-\frac{\begin{vmatrix}-1 & 3*2^2\\ 0 & -4*2^3\end{vmatrix}}{\begin{vmatrix}-1 & 3*2^2\\ 0 & -4*2^3\end{vmatrix}}=-1$$

analogicky $z\prime(0)=0$